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# A TOPOLOGICAL CHARACTERIZATION OF LINEARITY FOR QUASI-TRACES

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Abstract. Let $\mathcal{A}$ be a $C^*$-algebra, and let $\mu$ be a (local) quasi-trace on $\mathcal{A}$. Then $\mu$ is linear if, and only if, the restriction of $\mu$ to the closed unit ball of $\mathcal{A}$ is uniformly weakly continuous.

## Introduction

Throughout this paper, $\mathcal{A}$ will be a $C^*$-algebra and $A$ will be the real Banach space of selfadjoint elements of $\mathcal{A}$. The closed unit ball of $\mathcal{A}$ is $\mathcal{A}_1$ and the closed unit ball of $A$ is $A_1$. We do not assume the existence of a unit in $\mathcal{A}$.

**Definition.** A quasi-linear functional on $\mathcal{A}$ is a function $\mu : \mathcal{A} \to \mathbb{C}$ with the following properties:

(1) $\mu$ is bounded on $\mathcal{A}_1$.
(2) Whenever $a$ and $b$ are in $A$ then

$$\mu(a + ib) = \mu(a) + i\mu(b).$$

(3) Whenever $\mathcal{B}$ is an abelian $*$-subalgebra of $\mathcal{A}$ then the restriction of $\mu$ to $\mathcal{B}$ is linear.

Examples of non-linear, quasi-linear functionals on $M_2(\mathbb{C})$ exist in great abundance. See [7] for a full description of all the non-linear quasi-states on $M_2(\mathbb{C})$.

**Definition.** A quasi-linear functional, $\mu$, on $\mathcal{A}$ is said to be a *quasi-trace* if, whenever $x$ and $y$ are in $\mathcal{A}$,

$$\mu(xy) = \mu(yx).$$

We do *not* require a quasi-trace to be positive on positive elements of $A$.

Handelman [12], Blackadar and Handelman [4] and Haagerup [11] have made powerful contributions to the problem of determining when quasi-traces are linear. They consider bi-traces, that is, positive quasi-traces which are postulated to be well-behaved with respect to tensoring with $M_2(\mathbb{C})$, the algebra of two-by-two complex matrices.

In this note, we impose no algebraic restrictions on the quasi-traces considered. Instead, we show that for a quasi-trace, linearity is equivalent to uniform weak continuity on the unit ball.

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Our proofs are brief because most of the hard work has been done elsewhere. In particular, we proved in [6] that:

**Theorem.** *Let $\mathcal{A}$ be a $C^*$-algebra with no quotient isomorphic to $M_2(\mathbb{C})$. Let $\mu$ be a quasi-linear functional on $\mathcal{A}$. then $\mu$ is linear if, and only if, $\mu$ is uniformly weakly continuous on $\mathcal{A}_1$.*

This result is best possible, in the sense that, if $\mathcal{A}$ does have a quotient onto $M_2(\mathbb{C})$, then there exists a quasi-linear functional $\mu$ on $\mathcal{A}$, where $\mu$ is uniformly weakly continuous on $\mathcal{A}$, and $\mu$ is not linear; see [8]. However, when $\mu$ is a quasi-trace, we shall show that the analogue of the above theorem is true for *all* $\mathcal{A}$.

In fact, we shall use a weaker property than quasi-linearity, local quasi-linearity.

**Definition.** A function $\mu : \mathcal{A} \to \mathbb{C}$ is said to be *locally quasi-linear* if the following properties hold:

(1) $\mu$ is bounded on $\mathcal{A}_1$.

(2) Whenever $a$ and $b$ are in $A$ then

$$\mu(a + ib) = \mu(a) + i\mu(b).$$

(3) Whenever $\mathcal{B}$ is a closed $*$-subalgebra of $\mathcal{A}$ generated by a single Hermitian element then the restriction of $\mu$ to $\mathcal{B}$ is linear.

It is clear that when $\mu$ is quasi-linear, then $\mu$ is locally quasi-linear. Surprisingly, the converse is false, even when $\mathcal{A}$ is abelian and separable. Aarnes [2] constructs a counter-example where $\mathcal{A}$ is the $C^*$-algebra of continuous functions on the unit square. However, whenever $\mathcal{A}$ is sufficiently rich in projections, in particular when $\mathcal{A}$ is a von Neumann algebra, then local quasi-linearity does imply quasi-linearity [3].

**Definition.** A function $\mu : \mathcal{A} \to \mathbb{C}$ is a *local quasi-trace* if $\mu$ is locally quasi-linear and, for each $x$ and $y$ in $\mathcal{A}$,

$$\mu(xy) = \mu(yx).$$

## 1. Type $\mathrm{I}_n$ algebras

Let $S$ be a compact Hausdorff space, and let $C(S)$ be the commutative $C^*$-algebra of all continuous complex-valued functions on $S$. The selfadjoint part of $C(S)$ is a boundedly complete vector lattice if, and only if, $S$ is extremally disconnected, that is, the closure of each open subset of $S$ is also open. In all that follows we shall assume $S$ to be extremally disconnected.

Let $M_n(C(S))$ be the algebra of $n \times n$ matrices over $C(S)$. Then

$$M_n(C(S)) = C(S, M_n) = C(S) \otimes M_n.$$

We equip $M_n(C(S))$ with the natural involution and $C^*$-norm. The canonical centre-valued trace, $\Gamma$, on $M_n(C(S))$ is defined by

$$\Gamma[a_{ij}] = \frac{1}{n} \sum_{1}^{n} a_{rr}.$$

**Proposition 1.1.** *Let $\mu$ be a quasi-linear functional on $M_n(C(S))$ such that $\mu(UTU^*) = \mu T$ for each unitary $U$ and each selfadjoint $T$. Let $\bar{\mu}$ be defined on $C(S)$ by $\bar{\mu}(f) = \mu(f \otimes I)$. Then $\mu$ is linear and $\mu = \bar{\mu}\Gamma$.*

*Proof.* Let $\lambda = \bar{\mu}\Gamma - \mu$. Then $\lambda$ is quasi-linear and vanishes on the centre of $M_n(C(S))$. Moreover

$$\lambda(UTU^*) = \lambda(T)$$

for each unitary $U$ and selfadjoint $T$.

To simplify notation, we shall prove this proposition for $n = 2$, which is what we shall need in the next section. The proof for general $n$ is a trivial extension of the argument.

For $x \in C(S)$ we observe that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}.$$

So

$$\lambda \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} = \frac{1}{2}\lambda \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = 0.$$

Hence, for arbitrary $a$ and $b$ in $C(S)$,

$$\lambda \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \lambda \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = 0.$$

Let $T$ be a selfadjoint element of $M_2(C(S))$. Then (see Deckard and Pearcy [10]), there exists a unitary $U$, such that

$$UTU^* = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

where $a$ and $b$ are in $C(S)$. The $\lambda(T) = \lambda(UTU^*) = 0$. It follows that $\lambda$ vanishes on $M_2(C(S))$. So $\mu = \bar{\mu}\Gamma$ and hence $\mu$ is linear.

## 2. Locally quasi-linear traces

**Lemma 2.1.** *Let $\mathcal{M}$ be a von Neumann algebra, and let $\mu : \mathcal{M} \to \mathbb{C}$ be locally quasi-linear and such that $\mu(uxu^*) = \mu(x)$ for each unitary $u$ in $\mathcal{M}$ and each selfadjoint $x$ in $\mathcal{M}$. Then $\mu$ is linear.*

*Proof.* Let $\mathcal{B}$ be any maximal abelian $*$-subalgebra of $\mathcal{M}$. Then $\mathcal{B}$ is generated by its projections and so, by [3], the locally quasi-linear $\mu$ is linear on $\mathcal{B}$. Thus $\mu$ is quasi-linear on $\mathcal{M}$.

Let $\mathcal{M} = e\mathcal{M} \oplus (1-e)\mathcal{M}$ where $e$ is a central projection such that $e\mathcal{M}$ is of Type $\mathrm{I}_2$ and $(1-e)\mathcal{M}$ has no direct summand of Type $\mathrm{I}_2$.

For any selfadjoint $x$ in $\mathcal{M}$, since $e$ is central,

$$\mu(x) = \mu(ex) + \mu((1-e)x).$$

By Theorem B [5] or, in the case when $\mu \geq 0$ by Christensen [9] and Yeadon [12], the restriction of $\mu$ to $(1-e)\mathcal{M}$ is linear.

Since $e\mathcal{M}$ is of Type $\mathrm{I}_2$,

$$e\mathcal{M} \approx M_2(C(S))$$

where $S$ is extremally disconnected. (In fact, $S$ is hyperstonian.) So, by Proposition 1.1, the restriction of $\mu$ to $e\mathcal{M}$ is linear. Hence $\mu$ is linear on $\mathcal{M}$.

**Theorem 2.2.** *Let $\mu : \mathcal{A} \to \mathbb{C}$ be locally quasi-linear. Suppose that either*

(i) $\mu(xy) = \mu(yx)$ *for all $x$ and all $y$ in $\mathcal{A}$*

*or*

(ii) *$\mathcal{A}$ has a unit and, for each unitary $u$ and selfadjoint $x$, $\mu(uxu^*) = \mu(x)$.*

*Then $\mu$ is linear if, and only if, $\mu$ is uniformly continuous on $A_1$ with respect to the weak topology.*

*Proof.* By Lemma 1.1 [6], when $\mu$ is linear then $\mu$ is uniformly continuous on $\mathcal{A}_1$ with respect to the weak topology.

Let us now suppose that $\mu$ is uniformly weakly continuous on $\mathcal{A}_1$.

Let $H$ be the universal representation space of $\mathcal{A}$. We recall that $\mathcal{A}^{**}$, the second dual of $\mathcal{A}$, may be identified with the von Neumann envelope of $\mathcal{A}$ in $\mathcal{L}(H)$.

Then, by the proof of Theorem 3.1 [6], $\mu$ extends uniquely to a locally quasi-linear functional $\bar{\mu} : \mathcal{A}^{**} \to \mathbb{C}$, where $\bar{\mu}$ is continuous on $\mathcal{A}^{**}$ with respect to the strong operator topology of $\mathcal{L}(H)$.

We recall that $\mathcal{A}_1$ is strongly dense in the unit ball of $\mathcal{A}^{**}$. Hence, if (i) is satisfied, whenever $x$ and $y$ are in $\mathcal{A}^{**}$, there exist bounded nets in $\mathcal{A}$, $(x_i)$ and $(y_j)$, where $x_i \to x$ and $y_j \to y$ is the strong operator topology of $\mathcal{L}(H)$.

$$\bar{\mu}(x_i y_j) = \mu(x_i y_j) = \mu(y_j x_i) = \bar{\mu}(y_j x_i).$$

So

$$\bar{\mu}(x y_j) = \bar{\mu}(y_j x).$$

Hence

$$\bar{\mu}(xy) = \bar{\mu}(yx).$$

Alternatively, if (ii) is satisfied, then $1 \in \mathcal{A}$ and, for each unitary $u$ in $\mathcal{A}^{**}$, there is a net of unitaries $(u_j)$ in $\mathcal{A}$ which converges strongly to $u$ in $\mathcal{L}(H)$. Hence $\bar{\mu}(uxu^*) = \bar{\mu}(x)$ for each selfadjoint $x$ in $\mathcal{A}$. Hence $\bar{\mu}(uxu^*) = \bar{\mu}(x)$ for each selfadjoint $x$ in $\mathcal{A} * **$.

We may now apply Lemma 2.1 to deduce that $\bar{\mu}$ is linear on $\mathcal{A}^{**}$ and hence $\mu$ is linear on $\mathcal{A}$.

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