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ON THE PRIME MODEL PROPERTY

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ABSTRACT. Assume T is superstable, $\Phi(x)$ is a formula over $\emptyset$, $Q = \Phi(M^*)$ is countable and $K_Q = \{M : M$ is countable and $\Phi(M) = Q\}$. We investigate models in K_Q assuming K_Q has the prime model property. We prove some corollaries on the number of models in K_Q. We show an example of an ω-stable T and Q with K_Q having exactly 3 models.

First we fix the general set-up. Throughout, T is a countable complete theory in a first-order language L, $\Phi(x)$ is a formula of L without parameters, M^* is a countable model of T and $Q = \Phi(M^*)$. We work within a monster model $\mathcal{C} = \mathcal{C}^{eq}$ of T. All models of T we consider are elementary submodels of $\mathcal{C}$. Let $K_Q = \{M : M$ is countable and $\Phi(M) = Q\}$ and let $I(K_Q)$ be the number of models in K_Q, up to isomorphism.

The goal of this note is to develop some model theory of K_Q. As I pointed out in [Ne1, p.651], in general K_Q can not be treated with all common model-theoretic tools, since for example K_Q does not necessarily have the joint embedding property. It turns out however that if we assume just that K_Q has the prime model property (defined below), then it is much more manageable, at least for stable T. K_Q really may differ from an elementary class: I have found an example of K_Q with $I(K_Q) = 2$, while a theorem of Vaught [Sa] says that $I(T, \aleph_0) \neq 2$. In this example T is weakly minimal, but K_Q does not have the prime model property. In this paper we give an example of an ω-stable T with $I(T, \aleph_0) = \aleph_0$ and a strongly minimal Q with $I(K_Q) = 3$.

Now we recall some basic notions from [Ne1]. We call $f \in Aut(\mathcal{C})$ a Q-mapping if $f[Q] = Q$. For $a \subset M \in K_Q$ let $Aut_Q(\mathcal{C}/a)$ be the group of Q-mappings fixing a pointwise. This group acts in a Borel way on $S(Qa)$, inducing equivalence relation $E(a)$ on $S(Qa)$. The orbits of this action, that is the $E(a)$-classes, are called pseudotypes over Qa. When $a = \emptyset$, we use E to denote $E(a)$. Pseudotypes are Borel subsets of $S(Qa)$, and $E(a)$ is analytic. We say that $p \in S(Qa)$ is Q-isolated over a if $p/E(a)$ is not meager. $p/E(a)$ is the pseudotype of p over Qa (that is the equivalence class as a subset of $S(Qa)$, not a quotient of some sort). Q-isolation has many nice properties (see [Ne1, Ne2]), for instance if Q-isolated types are dense in $S(Q)$ then there is $M \in K_Q$ such that each $b \subset M$ satisfies a Q-isolated type over Q. Such an M is called Q-atomic. A Q-atomic model is unique up to isomorphism. Also, $tp(ab/Q)$ is Q-isolated iff $tp(a/Q)$ is Q-isolated and $tp(b/Qa)$ is Q-isolated over a.

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Let $\mathcal{B}(Q,a)=\{p\in S(Qa) :$ there is no $M\in K_Q$ containing a realization of $p\}$. $\mathcal{B}(Q,a)$ is a meager F_σ-subset of $S(Qa)$. $\mathcal{B}(Q)$ denotes $\mathcal{B}(Q,\emptyset)$. We think of the types in $\mathcal{B}(Q)$ as 'bad' types, since they can not be used to build models in K_Q. Types in $S(Q)\setminus\mathcal{B}(Q)$ are called good.

We say that K_Q has the prime model property if the following holds.

(PM) For every a with $tp(a/Q)$ good there is a model $M\in K_Q$ containing a, such that for every $M'\in K_Q$ containing a, some Q-mapping fixing a embeds M into M'.

The model M occurring in (PM) is called Q-prime over a. 'Q-prime' means Q-prime over $\emptyset$.

Remark. (1) If M is Q-prime over a then M is Q-atomic over a. In particular, a Q-prime over a model is unique up to isomorphism over a. (However, even a countable Q-atomic model need not be Q-prime.)

(2) If (PM) holds and $p\in S(Qa)$ then p is Q-isolated over a iff every $M\in K_Q$ containing a contains a realization of a type $p'\in p/E(a)$.

(3) (PM) implies that Q-isolated over a types are dense in $S(Qa)$ and "Q-prime over a " ="Q-atomic over a ".

Proof. (1) Wlog $a=\emptyset$. We must show that if $p\in S(Q)$ is not Q-isolated then no $p'\in p/E$ is realized in M. To prove this it is enough to show that some $N\in K_Q$ omits every $p'\in p/E$. Since p is not Q-isolated, p/E is meager, hence is covered by countably many closed nowhere dense subsets $X_n, n<\omega$, of $S(Q)$. A closed subset X_n of $S(Q)$ corresponds to a type q_n over Q, and X_n being nowhere dense means that q_n is non-isolated. Hence by the omitting types theorem there is a model $N\in K_Q$ omitting every $q_n, n<\omega$. This means that N omits every $p'\in p/E$. (After all, the omitting types theorem is just the Baire category theorem adapted for the needs of model theorists.) Since a Q-atomic model is unique up to isomorphism, also a Q-prime model is such.

(2),(3) follow easily.

We say that K_Q has the joint embedding property if the following holds.

(JE) For every a with $tp(a/Q)$ good, if $M_0,M_1\in K_Q$ contain a then for some $M\in K_Q$ containing a there are Q-mappings f_0,f_1 fixing a and embedding M_0,M_1 respectively into M.

We say that $M\in K_Q$ is Q-saturated if for every $a\subset M$ and good $p\in S(Qa)$, some type in $p/E(a)$ is realized in M. $M\in K_Q$ is Q-universal if for every $N\in K_Q$ there is a Q-mapping embedding N into M.

Lemma 1. *(1) If T is stable, there are countably many good pseudotypes over Q and (PM) holds, then there is a Q-saturated model.*

(2) A Q-saturated model is Q-universal and unique up to isomorphism.

(3) If (T is stable and (PM) holds) or (a Q-saturated model exists) then (JE) holds.

Proof. [Ne4, Corollary 2.10(3)] proves that if T is stable and (PM) holds, then whenever the types $tp(a/Q),tp(b/Q)$ are good, then for some b' with $tp(b'/Q)\,E\,tp(b/Q)$, the type $tp(ab'/Q)$ is good. This amalgamation property easily implies the lemma.

The following fact ([Ne4, Corollary 2.10]) shows that the assumption of (PM) makes sense.

Fact. *Assume T is small, stable, $T\lceil\Phi$ is ω-stable and one of the following conditions holds.*

(i) $T\lceil\Phi$ is bounded, 1-based or of finite rank, and $I(T\lceil\Phi,\aleph_0)<2^{\aleph_0}$.

(ii) Q is atomic or saturated (as a model of $T\lceil\Phi$).

(iii) T is superstable and $T\lceil\Phi$ has $<2^{\aleph_0}$ countable models.

Then K_Q has the prime model property and there are countably many good pseudotypes over Q.

It would be very good if we could omit the assumption in (i) that $T\lceil\Phi$ is bounded, 1-based or of finite rank. Then we could drop (iii) in the Fact. It is open whether T stable, $T\lceil\Phi$ ω-stable and $I(T,\aleph_0)<2^{\aleph_0}$ implies (PM). This is related to the τ-stability conjecture [Ne2, Ne4]. As I mentioned above, there is an example of a weakly minimal T and K_Q with $I(K_Q)=2$. In this example K_Q does not have the prime model property. Now we give an example of an ω-stable T with $I(T,\aleph_0)=\aleph_0$ and a strongly minimal Q with $I(K_Q)=3$.

Example. The example is a modification of Shelah's example of a theory T_0 with abnormal types ([Ba, XVIII,4]). Let V be a countably infinite set and $P=\{c_n,n<\omega\}$ an infinite and co-infinite subset of V. Equip $V\setminus P$ with a structure of a model of T_0. This means among others that there is an equivalence relation E' on $V\setminus P$ and an asymmetric relation R on $(V\setminus P)/E'$ such that $((V\setminus P)/E';R)$ is a directed graph without cycles such that each element has infinitely many successors and predecessors (see [Ba] for details). We can arrange that $((V\setminus P)/E';R)$ is connected. So on $(V\setminus P)/E'$ there is a natural distance function $d(x,y)$. Define a ternary relation $S\subset P\times(V\setminus P)/E'\times(V\setminus P)/E'$ by: $S(c_n,x,y)$ iff $d(x,y)\leq n$.

Let $M^*=(V;P,S,c_n,n<\omega$; the structure of $V\setminus P$ as a model of $T_0)$, and $T=Th(M^*)$. Clearly, T is ω-stable, $I(T,\aleph_0)=\aleph_0$, and $\Phi(x)=P(x)$ is trivial and strongly minimal. Let M' be a prime model of T and $Q=P(M')$. Then for any $M\in K_Q$, $\neg P(M)/E$ is R-connected. By [Ba, XVIII,4.6] in T_0, up to isomorphism there are 3 kinds of countable connected components. Hence $I(K_Q)=3$. ω-stability implies K_Q has the prime model property, hence by Theorem 1 below we can not have here $I(K_Q)=2$. However, if M is a countable model of T with $P(M)\neq\{c_n,n<\omega\}$, then for $Q=\Phi(M)$, $I(K_Q)$ is infinite.

In this paper we try to generalize a result of Lachlan [Ba] saying that for superstable T , $I(T,\aleph_0)>1$ implies $I(T,\aleph_0)\geq\aleph_0$. We shall use the following lemma.

Lemma 2 ([Ne1, Lemma 2.4])**.** *If T is stable, $tp(a/Qb)$ is Q-isolated over b and $tp(a/Q)$ is not Q-isolated, then $a\not\perp b(Q)$.*

We write $a\perp b(X)$ for 'a is independent from b over X'. Regarding the theorem of Vaught we can prove the following.

Theorem 1. *If T is superstable and K_Q has the prime model property, then $I(K_Q)\neq 2$. If moreover T has finite U-rank and $I(K_Q)>1$, then $I(K_Q)$ is infinite.*

Proof. Suppose $I(K_Q)>1$. If every good type in $S_n(Q),n<\omega$, is Q-isolated, then every model in K_Q is Q-atomic, hence $I(K_Q)=1$, a contradiction. So let $p\in S(Q)\setminus\mathcal{B}(Q)$ be non-Q-isolated. Let M_0 be the Q-prime model.

If $(S_n(Q)\setminus\mathcal{B}(Q))/E$ is uncountable, then clearly $I(K_Q)$ is infinite. So we can assume there are only countably many good pseudotypes over Q. By Lemma 1(1) we see that there is a Q-saturated model M_ω. Since M_ω realizes a non-Q-isolated type, $M_0\not\cong M_\omega$.

Let a realize p and let M be Q-prime over a. Wlog $M_0 \subseteq M$. Since p is not Q-isolated, $a \in M \setminus M_0$. Since $Q = \Phi(M) = \Phi(M_0)$, M and M_0 are a Vaughtian pair. In particular, $tp(a/M_0)$ is non-algebraic and orthogonal to Φ. Choose a finite $b \subset M_0$ with $a \downarrow M_0(b)$. So also $tp(a/Qb) \perp \Phi$. Also, $tp(a/Qb)$ is not Q-isolated over b. Otherwise, since $tp(b/Q)$ is Q-isolated (as $b \subset M_0$), by transitivity of Q-isolation ([Ne1]) we would get that $p = tp(a/Q)$ is Q-isolated.

Let $I = \{a_n, n < \omega\}$ be a Morley sequence in $stp(a/Qb)$, with $a = a_0$. Wlog $I \subseteq M_\omega$. By superstability, for some n', if $n > n'$ then $a_n \downarrow Qb(a_{<n})$ and $tp(a_n/a_{<n})$ is stationary. Hence for $n > n'$ also $tp(a_n/Qba_{<n})$ is not Q-isolated over $ba_{<n}$. Indeed, since $tp(a_n/ba_{<n})$ is orthogonal to Φ, by the open mapping theorem it is enough to show that $tp(a_n/ba_{<n})$ is non-isolated. Suppose otherwise. Since $a \downarrow M_0(b)$ the construction yields $a_n \downarrow a_{<n}(b)$. This gives that $tp(a_n/b) = tp(a/b)$ is isolated and stationary, because $tp(a_n/ba_{<n})$ is stationary. Hence $tp(a/b) \vdash tp(a/Qb)$, and $tp(a/Qb)$ is isolated, hence Q-isolated over b, a contradiction.

The rest of the proof resembles that of [Ne1, Theorem 2.5]. Let M_n be Q-prime over $ba_{<n}$. Clearly for $n > 0$, $M_n \ncong M_0$, since p is realized in M_n. Let $m > n'$ and suppose M_m and M_ω are isomorphic. Then within M_m there is a copy $b'\{a'_n, n < \omega\}$ of bI (via some Q-mapping). By the properties of Q-isolation, for every $c \subset M_m$, M_m is Q-atomic over $ba_{<m}c$. Hence M_m is Q-atomic over $ba_{<m}b'a'_{<n}$ for any n. In particular, for every $n > n'$ we have that $tp(a'_n/Qba_{<m}b'a'_{<n})$ is Q-isolated over $ba_{<m}b'a'_{<n}$ and $tp(a'_n/Qb'a'_{<n})$ is not Q-isolated over $b'a'_{<n}$. By Lemma 2 we get that for every $n > n'$, $ba_{<m} \not\downarrow a'_n(Qb'a'_{<n})$, contradicting the superstability of T.

The case when U-rank is finite is handled like in [Ne1]. We prove that if k is large enough then $M_k \ncong M_m$.

I managed to generalize Lachlan's theorem for K_Q in case when Q is weakly saturated (as a model of $T\lceil\Phi$), that is when every type in $T\lceil\Phi$ is realized in Q.

Theorem 2. *If T is superstable, K_Q has the prime model property, Q is weakly saturated and $I(K_Q) > 1$, then $I(K_Q)$ is infinite.*

The proof relies on the following lemma.

Lemma 3. *If T is superstable and Q is weakly saturated, then every type in $S(\emptyset)$ has a good extension in $S(Q)$.*

Proof. Choose any a. We want to find $a' \equiv a$ with $tp(a'/Q)$ good. By superstability, for some $b \subset \Phi(\mathcal{C})$ we have $a \downarrow \Phi(\mathcal{C})(b)$. This implies that if $b \subset Q' = \Phi(M')$ for some countable M', then $tp(a/Q') \notin \mathcal{B}(Q')$. Indeed, otherwise for some $d \in \Phi(\mathcal{C})$ we would have $a \not\downarrow d(Q')$, while $a \downarrow \Phi(\mathcal{C})(Q')$, a contradiction.

Since Q is weakly saturated, there is $b' \subset Q$ with $b \equiv b'$. Choose a' with $a'b' \equiv ab$. It follows that $a' \downarrow \Phi(\mathcal{C})(b')$ and $tp(a'/Q)$ is good.

Note that if in Lemma 3 we weaken the assumption of superstability to stability, then we must assume that Q is universal.

Proof of Theorem 2. As in the proof of Theorem 1 we find a, b with $tp(a/Qb)$ non-Q-isolated over b and orthogonal to Φ (hence good). We can also assume that $a \downarrow Q(b)$ and $tp(a/b)$ is stationary non-isolated (see the proof of Theorem 1). Let $k = w(ab)$.

By Lemma 3, for each $n < \omega$ there is a Morley sequence A_n in $stp(ab)$ of length $(k^{n+1}-1)/(k-1)$ such that $tp(A_n/Q)$ is good. Let M_n be Q-prime over A_n. Let $|A_n|$ denote the length of A_n and let $w(A_n)$ be the weight of A_n. Notice that

$$w(A_n) = k\frac{k^{n+1}-1}{k-1} = |A_{n+1}| - 1$$

Hence for $n < m,\ w(A_n) < |A_m|$.

We shall prove that for $n < m,\ M_n$ and M_m are non-isomorphic. Suppose not. Then there is an $A'_m \subset M_n$, a Morley sequence in $stp(ab)$ of length $|A_m|$. Since $w(A_n) < |A_m|$, for some $a'b' \in A'_m,\ a'b' \downarrow A_n$. In particular, $a' \downarrow A_n(b')$. Since $tp(a'/b')$ is orthogonal to Φ, also $a' \downarrow A_n Q(b')$. As M_n is Q-atomic over A_n, M_n is also Q-atomic over $A_n b'$. Hence $tp(a'/A_n Q b')$ is Q-isolated over $A_n b'$. As $tp(a'/b')$ is stationary, non-isolated and orthogonal to Φ, $tp(a'/Qb')$ is not Q-isolated over b'. By Lemma 2, $a' \not\downarrow A_n(Qb')$, a contradiction.

Corollary 1. *Assume T is superstable, $T\lceil\Phi$ is ω-stable and Q is saturated. Then $I(K_Q) > 1$ implies $I(K_Q)$ is infinite. On the other hand, $I(K_Q) = 1$ implies T is ω-stable and (ω,ω)-categorical relative to Φ (in the sense of* [HHM]*).*

Proof. If T is not small, then Lemma 3 implies that $I(K_Q)$ is infinite. So assume T is small. By the Fact, K_Q has the prime model property. Hence by Theorem 2, $I(K_Q) > 1$ implies $I(K_Q)$ is infinite.

Now assume $I(K_Q) = 1$. If T is not ω-stable then for some a there is an isolated type $p \in S(a)$ without Morley rank, such that each type of smaller ∞-rank has Morley rank (see e.g. [Ne3]). It follows that p has infinite multiplicity, hence if b realizes p then $p' = p|ab$ is non-isolated. By Lemma 3, we can assume that $tp(ab/Q)$ is good. Since $p' \perp \Phi$, we have $p' \vdash p''$ for some $p'' \in S(Qab)$. Clearly p'' is not Q-isolated over ab. Hence $I(K_Q) > 1$, a contradiction.

So T is ω-stable. It follows that the only model in K_Q is prime over Q in the usual sense.

Now suppose $Q' = \Phi(M')$ is countable. Wlog $Q' \subseteq Q$. If some $p \in S(Q')$ is non-isolated and good then also $p' = p|Q$ is non-isolated and good. p' is realized in a model in K_Q which is not prime over Q, a contradiction. We see that every model in $K_{Q'}$ is prime over Q'. Hence T is (ω,ω)-categorical relative to Φ in the sense of [HHM].

I would like to stress the fact, implicit in the proof of Theorem 2, that the more ample Q is, the larger $I(K_Q)$ is. To make this more explicit, we prove the following corollary, which deals with the case of strongly minimal Φ, when we have a good notion of dimension of Q.

Corollary 2. *Assume T is small superstable, Φ is strongly minimal and $I(K_Q) > 1$. Then there are $k, l < \omega$ such that if $Q' = \Phi(M')$ is countable, $n > 2$ and*

$$dim(Q') \geq \frac{k^n - k}{k-1} + l$$

then $I(K_{Q'}) \geq n$.

Proof. By the Fact, $K_{Q'}$ has the prime model property. Also we can assume that there are countably many good pseudotypes over Q', hence there is a Q'-saturated model. Wlog the strongly minimal type $r \in S(\emptyset)$ containing Φ is eventually non-isolated. Let l be the largest number such that r^l is isolated. First we prove the following variant of Lemma 3.

(a) If $p \in S(\emptyset)$ and $dim(Q') \geq w(p)+l$ then there is $p' \in S(Q') \setminus \mathcal{B}(Q')$ extending p.

Indeed, let a realize p and $C = Cb(a/\Phi(\mathcal{C}))$. Since $C \subseteq acl(a)$, $w(C) \leq w(a)$. Choose a Morley sequence I in r of size l, with $I \perp C$, and let Q'' be a model of $T\lceil\Phi$ prime over CI. Clearly $w(C/I) = w(C)$. If $d \in Q''$ realizes $r|I$ then $tp(d/CI)$ is isolated and $tp(d/I)$ is non-isolated, hence $d \not\perp C(I)$. Thus $dim(r|I, Q'') \leq w(C)$, and $dim(Q'') \leq w(C) + l \leq dim(Q)$. In particular, Q'' may be embedded into Q'. Now (a) follows as in Lemma 3.

Now we find a, b as in the proof of Theorem 2. Let $k = w(ab)$. Hence $w(A_n) = (k^{n+2} - k)/(k-1)$. If

$$dim(Q') \geq \frac{k^{n+2} - k}{k-1} + l$$

then by (a), for $m \leq n$ we can find $A'_m \equiv A_m$ with $tp(A'_m/Q')$ good.

Let M_0 be Q'-prime, for $0 < m \leq n$ let M_m be Q'-prime over A'_m and let M_{n+1} be Q'-saturated. The proof of Theorem 2 shows that $M_m, m \leq n$, are non-isomorphic. The proof of Theorem 1 shows they are also non-isomorphic to M_{n+1}. Hence $I(K_{Q'}) \geq n+2$.

We showed above an example of an ω-stable T and Q with $I(K_Q) = 3$. It may be interesting to point that any such example should resemble the classical situation.

Proposition. *Assume T is superstable, K_Q has the prime model property and $I(K_Q) = 3$. Then the 3 models in K_Q are: the Q-prime one, the Q-saturated one and the third model M which is neither Q-prime nor Q-saturated. M is characterized by the following condition.*

(*) *For every a with $tp(a/Q)$ good and not Q-isolated, M is isomorphic to a model Q-prime over a.*

Proof. First we prove that M satisfies (*). Suppose $tp(a/Q)$ is good and not Q-isolated, and let N be Q-prime over a. So N is not Q-prime. The proof of Theorem 1 shows N is not Q-saturated. Hence N is isomorphic to M. This shows (*). The other direction is equally easy.

Most of the results of this note may be adapted to the situation when Q is fixed pointwise by $Aut(\mathcal{C})$, that is when the elements of Q are named by constants of the language.

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