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THE ZEROS OF THE FIRST TWO DERIVATIVES OF A MEROMORPHIC FUNCTION

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Abstract. We prove a theorem which implies the following: if f is meromorphic of finite order in the plane and f' and f'' have only finitely many zeros, then f has only finitely many poles.

1. Introduction

We begin with the following theorem, using the term meromorphic henceforth to mean meromorphic in the plane.

Theorem A. *Suppose that f is meromorphic and that f and $f^{(k)}$ have only finitely many zeros, for some $k \geq 2$. Then f has the form $f(z) = R(z)\exp(P(z))$, with R rational and P a polynomial. In particular, f has finite order and only finitely many poles.*

Theorem A was conjectured by Hayman in 1959 [9], [10], [11], and was proved for $k \geq 3$ by Frank [6]. The case $k = 2$ was settled in [13], having been proved by Mues [16] for functions of finite lower order. Simple examples show that Theorem A is not true for $k = 1$ (see, however, [4]). Now, it is easy to give examples of entire functions f such that f'' has no zeros, but the following conjecture seems reasonable.

Conjecture 1. *If f is meromorphic of finite order and f'' has only finitely many zeros, then f has only finitely many poles.*

We remark that (see Satz 5 of [15]), for any $k \geq 2$ and for any transcendental meromorphic function f, setting $L = -f^{(k+1)}/(k+1)f^{(k)}$ and applying the first fundamental theorem to $L' + L^2$ gives

$$(1) \qquad (k-1)N_1(r,f) \leq 2\overline{N}_2(r,f) + 2\overline{N}(r,1/f^{(k)}) + O(\log rT(r,f^{(k)}))$$

as r tends to infinity outside a set of finite measure. Here $N_1(r,f)$ counts the simple poles of f, while $\overline{N}_2(r,f)$ counts the points at which f has multiple poles (see [10], [12] for notation). Thus Conjecture 1 is true if f has at most finitely many multiple poles. We remark further that it has been conjectured by Gol'dberg that $\overline{N}(r,f)$ may be estimated in terms of $N(r,1/f'')$ and an error term which is $o(T(r,f))$, and we refer the reader to [7], [8], [15], [18], in particular with regard to the related Mues conjecture that $\sum_{a\in\mathbf{C}}\delta(a,f') \leq 1$.

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We note further that Conjecture 1 is false for f of infinite order, as the following construction shows. Let Π be any entire function having infinitely many zeros, all simple, and use the Mittag-Leffler theorem to construct an entire function h such that $g(z) = \Pi^{-1} e^h$ has Laurent series development $g(z) = -2/(z-a) + O(|z-a|)$ near each zero a of Π. It is then easy to see that $f''/f' = g$ defines a meromorphic function such that f' and f'' have no zeros. For f of finite order, no such examples are possible.

Theorem 1. *Let f be meromorphic of finite order such that f' and f'' have only finitely many zeros. Then f''/f' is rational and, in particular, f has only finitely many poles.*

Theorem 1 will be deduced from a result concerning the class B, which consists of those meromorphic functions f such that the set of finite singularities of the inverse function f^{-1} is bounded. This means that there is some $S > 0$ such that f has no critical values or asymptotic values w with $S < |w| < \infty$. The following lemma was proved by Eremenko and Lyubich [5] (see also [1], [2]).

Lemma B. *Let f be transcendental and meromorphic, in the class B, with $f(0) \neq \infty$. Then there exist positive constants c, R such that we have, for all z, the estimate $|zf'(z)/f(z)| \geq c \ \log^+ |f(z)/R|$.*

Lemma B is proved by noting that there exists $R > 0$ such that $|f(z)| < R$ on a path from 0 to ∞ and then, provided R is large enough, applying Bloch's theorem to the function $\log(f^{-1}(e^w))$ in $Re(w) > \log R$. We shall prove the following:

Theorem 2. *Let f be transcendental meromorphic, in the class B, such that f''/f' has only finitely many zeros. Then f''/f' is rational.*

The assumption in Theorem 2 that f is in the class B is not redundant, as the examples $f(z) = z - \tan z$, $g(z) = \int_0^z \int_0^t \exp(s^2) ds dt$ show.

Corollary. *Let f be transcendental and meromorphic of finite order, with only finitely many critical values, and suppose that f''/f' has only finitely many zeros. Then f''/f' is rational.*

This corollary obviously establishes Theorem 1, and itself follows at once from Theorem 2 because, if f has finite order and only finitely many critical values, a recent result of Bergweiler and Eremenko [3] implies that f has only finitely many asymptotic values.

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2. Proof of Theorem 2

We assume the existence of a meromorphic function f in the class B such that f''/f' is transcendental but has only finitely many zeros. We may clearly assume that $f(0)$ is finite. We note that f'/f'' has only finitely many poles, and we set $u(z) = \log |f'(z)/f''(z)|$.

By a result of Lewis, Rossi and Weitsman [14], there is a path Γ, starting at z_0, say, and tending to infinity, such that $u(z)/(\log |z|) \to +\infty$ as $z \to \infty$ on Γ, while the part of Γ joining z_0 to z has length at most $\exp(o(u(z)))$. This length estimate may be found as (3.7) of [14] and is stated explicitly in [17]. We parametrize Γ with respect to arc length and, for n a large positive integer, define s_n by $s_n = \sup\{s : u(\Gamma(s)) \leq n\}$. If n is large and $s_n \leq s < s_{n+1}$ we have

$$\left|\int_{\Gamma(s)}^{\infty}(f''(z)/f'(z))dz\right| \leq \sum_{k=n}^{\infty} e^{-k}e^{o(k+1)} \leq c_1 e^{-n/2}, \tag{2}$$

using c_j to denote positive constants. Thus f' tends to a finite non-zero value b as z tends to infinity on Γ, and we can assume without loss of generality that $b = 1$. Now we have, for n large and $s_n \leq s < s_{n+1}$, the estimate $|f'(\Gamma(s)) - 1| \leq c_2|\log|f'(\Gamma(s))|| \leq e^{-n/4}$ and so

$$\left|\int_{\Gamma(s)}^{\infty}(f'(z) - 1)dz\right| \leq \sum_{k=n}^{\infty} e^{-k/4}e^{o(k+1)} = o(1). \tag{3}$$

Thus $f(z) = z + O(1)$ and $zf'(z)/f(z) = 1 + o(1)$ as z tends to infinity on Γ, which plainly contradicts Lemma B and proves Theorem 2.

We remark finally that a modification of the above proof shows that Theorem 2 holds with f''/f' replaced by $f^{(k+1)}/f^{(k)}$, for any $k \geq 2$.

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