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STABILITY AND DICHOTOMY OF POSITIVE SEMIGROUPS ON L_p

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ABSTRACT. A new proof of a result of Lutz Weis is given, that states that the stability of a positive strongly continuous semigroup $(e^{tA})_{t\geq 0}$ on L_p may be determined by the quantity $s(A)$. We also give an example to show that the dichotomy of the semigroup may not always be determined by the spectrum $\sigma(A)$.

Consider a strongly continuous semigroup $(e^{tA})_{t\geq 0}$ acting on a Banach space X with unbounded generator A. It has long been known that the spectral mapping theorem $e^{t\sigma(A)} = \sigma(e^{tA}) \setminus \{0\}$ does not necessarily hold. (Here $\sigma(A)$ denotes the spectrum of an operator A.) Indeed, let $s(A) = \sup \mathrm{Re}(\sigma(A))$, and let $\omega(A) = \sup \mathrm{Re}(\log(\sigma(e^A))) = \inf\{\lambda : \left\|e^{tA}\right\| \leq M_\lambda e^{\lambda t}\}$. Then there are examples of semigroups for which $s(A) \neq \omega(A)$ (see [N]).

The purpose of this paper is to give one situation in which it is true that $s(A) = \omega(A)$. This next result has already been proved by Lutz Weis [We]. We will give a different, shorter proof. We refer the reader to [We] for a history of the problem.

Theorem 1. *Let e^{tA} be a strongly continuous positive semigroup on $L_p(\Omega, \mathcal{F}, \mu)$, where $(\Omega, \mathcal{F}, \mu)$ is a sigma-finite measure space and $1 \leq p < \infty$. Then $\omega(A) = s(A)$.*

In order to show this result, we will make use of the following lemmas. The first result may be derived from [C], Theorem 7.4 (the reader may like to know that a proof of the 'Pringsheim-Landau Theorem' used in [C] may be found on page 59 of [Wi]).

Lemma 2. *Let e^{tA} be a strongly continuous positive semigroup on a Banach lattice X, and let $g \in X$. Then for any $\lambda > s(A)$ we have that*

$$(\lambda - A)^{-1} g = \int_0^\infty e^{s(A-\lambda)} g \, ds.$$

Here the right-hand side is taken in the sense of an improper integral.

The next result may be found in [LM1] and [LM2].

Lemma 3. *Let e^{tA} be a strongly continuous semigroup on a Banach space X, and let $1 \leq p < \infty$. Then $1 \notin \sigma(e^{2\pi A})$ if and only if $i\mathbb{Z} \cap \sigma(A) = \emptyset$ and there is a*

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constant $c > 0$ such that for any $v_{-n}, v_{-n+1}, \ldots, v_n \in X$ we have

$$\int_0^{2\pi} \left\| \sum_{k=-n}^{n} (ik - A)^{-1} v_k e^{ikt} \right\|^p dt \leq c^p \int_0^{2\pi} \left\| \sum_{k=-n}^{n} v_k e^{ikt} \right\|^p dt.$$

For the next result, we specialize to a Banach lattice of functions on a sigma-finite measure space. In fact, this is really no loss of generality, and the interested reader should find no trouble making sense of this result for a general Banach lattice by applying the ideas in [LT], Chapter 1.4.

Lemma 4. *Let P be a positive operator on X, a Banach lattice of functions on a sigma-finite measure space, such that $|g| \leq f \in X$ implies that $g \in X$. Let $1 \leq p < \infty$. If $f : [0, 2\pi] \to X$ is a measurable, simple function, then*

$$\left(\int_0^{2\pi} |P(f(t))|^p \, dt \right)^{1/p} \leq P \left(\left(\int_0^{2\pi} |f(t)|^p \, dt \right)^{1/p} \right).$$

Proof. Let us set $f = \sum_{k=1}^n v_k \chi_{A_k}$, where $v_k \in X$ and the sets $A_k \subseteq [0, 2\pi]$ are disjoint. Then, letting $f_k = v_k \, |A_k|^{1/p}$, the result reduces to showing that

$$\left(\sum_{k=1}^n |P(f_k)|^p \right)^{1/p} \leq P \left(\left(\sum_{k=1}^n |f_k|^p \right)^{1/p} \right).$$

However, we know that

$$\left(\sum_{k=1}^n |f_k|^p \right)^{1/p} = \underset{\sum |a_k|^q \leq 1}{\text{l.u.b.}} \sum_{k=1}^n \mathrm{Re}(a_k f_k).$$

Here, l.u.b. denotes the least upper bound in the lattice. Now, since P is positive, we have that

$$P \left(\underset{\sum |a_k|^q \leq 1}{\text{l.u.b.}} \sum_{k=1}^n \mathrm{Re}(a_k f_k) \right)$$

is an upper bound for $\sum_{k=1}^n \mathrm{Re}(a_k P(f_k))$ whenever $\sum |a_k|^q \leq 1$. Hence

$$\begin{aligned} \left(\sum_{k=1}^n |P(f_k)|^p \right)^{1/p} &= \underset{\sum |a_k|^q \leq 1}{\text{l.u.b.}} \sum_{k=1}^n \mathrm{Re}(a_k P(f_k)) \\ &\leq P \left(\underset{\sum |a_k|^q \leq 1}{\text{l.u.b.}} \sum_{k=1}^n \mathrm{Re}(a_k f_k) \right) \\ &= P \left(\left(\sum_{k=1}^n |f_k|^p \right)^{1/p} \right). \end{aligned}$$

Proof of Theorem 1. It is well known that $s(A) \leq \omega(A)$ (see [N]). Thus by simple rescaling arguments, we see that it is sufficient to show that if $s(A) < 0$, then $\mathbb{T} \cap \sigma(e^{2\pi A}) = \emptyset$.

We will show, under the assumption that $s(A) < 0$, that if $f : \mathbb{R} \to L_p$ is a bounded, measurable function that is periodic with period 2π, then for each $N > 0$ we have

$$\left(\int_0^{2\pi} \left\| \int_0^N e^{sA} f(t-s)\, ds \right\|_{L_p}^p dt \right)^{1/p} \le \left\| A^{-1} \right\| \left(\int_0^{2\pi} \| f(t) \|_{L_p}^p \, dt \right)^{1/p} .$$

In order to show this, we may assume without loss of generality that f restricted to $[0, 2\pi]$ is a simple function. Then $(s,t) \mapsto e^{sA} f(t)$ is Bochner measurable, and hence, without loss of generality, we may suppose that the map $(s,t,\omega) \mapsto [e^{sA} f(t)](\omega)$ is jointly measurable. (Recall that strictly speaking, an element of L_p is an equivalence class of functions, and so here we are making a choice.) Fix $N > 0$. By the positivity of e^{sA} and Fubini's Theorem, we have that

$$\begin{aligned}\left(\int_0^{2\pi} \left\| \int_0^N e^{sA} f(t-s)\, ds \right\|_{L_p}^p dt \right)^{1/p} &\le \left(\int_0^{2\pi} \left\| \int_0^N e^{sA} \left| f(t-s) \right| \, ds \right\|_{L_p}^p dt \right)^{1/p} \\ &= \left\| \left(\int_0^{2\pi} \left(\int_0^N e^{sA} \left| f(t-s) \right| \, ds \right)^p dt \right)^{1/p} \right\|_{L_p} .\end{aligned}$$

By the integral version of Minkowski's Theorem (see [HLP], Section 203), it follows that for each $\omega \in \Omega$

$$\begin{aligned}&\left(\int_0^{2\pi} \left(\int_0^N [e^{sA} \left| f(t-s) \right|](\omega)\, ds \right)^p dt \right)^{1/p} \\ &\qquad \le \int_0^N \left(\int_0^{2\pi} \left([e^{sA} \left| f(t-s) \right|](\omega) \right)^p dt \right)^{1/p} ds \\ &\qquad = \int_0^N \left(\int_0^{2\pi} \left([e^{sA} \left| f(t) \right|](\omega) \right)^p dt \right)^{1/p} ds.\end{aligned}$$

Finally, from Lemma 4, we see that

$$\left(\int_0^{2\pi} (e^{sA} \left| f(t) \right|)^p \, dt \right)^{1/p} \le e^{sA} \left(\int_0^{2\pi} \left| f(t) \right|^p \, dt \right)^{1/p} .$$

Putting all of these together, and applying Lemma 2, we obtain

$$\begin{aligned}\left(\int_0^{2\pi} \left\| \int_0^N e^{sA} f(t-s)\, ds \right\|_{L_p}^p dt \right)^{1/p} &\le \left\| \int_0^N e^{sA} \left(\int_0^{2\pi} \left| f(t) \right|^p dt \right)^{1/p} ds \right\|_{L_p} \\ \le \left\| \int_0^\infty e^{sA} \left(\int_0^{2\pi} \left| f(t) \right|^p dt \right)^{1/p} ds \right\|_{L_p} &= \left\| A^{-1} \left(\int_0^{2\pi} \left| f(t) \right|^p dt \right)^{1/p} \right\|_{L_p} \\ \le \left\| A^{-1} \right\| \left\| \left(\int_0^{2\pi} \left| f(t) \right|^p dt \right)^{1/p} \right\|_{L_p} &= \left\| A^{-1} \right\| \left(\int_0^{2\pi} \| f(t) \|_{L_p}^p \, dt \right)^{1/p} ,\end{aligned}$$

where the last equality uses Fubini's theorem. Now, if $f(t) = e^{i\beta t} \sum_{k=-n}^{n} v_k e^{ikt}$ for some $\beta \in \mathbb{R}$, then by Lemma 2, we see that

$$\int_0^N e^{sA} f(t-s)\, ds \to \sum_{k=-n}^{n} (ik + i\beta - A)^{-1} v_k e^{ikt}$$

uniformly in t as $N \to \infty$. Hence by Lemma 3 it follows that $e^{i\beta} \notin \sigma(e^{2\pi A})$.

One might conjecture that the spectrum of the generator of a positive semigroup e^{tD} on an L_p space might characterize the dichotomy of the semigroup, that is, if a is any real number, then $(a + i\mathbb{R}) \cap \sigma(D) = \emptyset$ if and only if $e^{ta}\mathbb{T} \cap \sigma(e^{tD}) = \emptyset$. However, this is not the case, as the next result shows.

Theorem 5. *There is a positive semigroup e^{tD} acting on an L_2 space such that $(1 + i\mathbb{R}) \cap \sigma(D) = \emptyset$, but $e^{2\pi} \in \sigma(e^{2\pi D})$.*

Proof. For each $M \in \mathbb{N}$, let C_M be the contraction acting on ℓ_2^M by the matrix

$$C_M = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Note that if $\lambda \neq 0$, then

$$(\lambda - C_M)^{-1} = \sum_{j=0}^{M-1} \lambda^{-1-j} C_M^j = \begin{bmatrix} \lambda^{-1} & \lambda^{-2} & \lambda^{-3} & \lambda^{-4} & \cdots & \lambda^{-M} \\ 0 & \lambda^{-1} & \lambda^{-2} & \lambda^{-3} & \cdots & \lambda^{-M+1} \\ 0 & 0 & \lambda^{-1} & \lambda^{-2} & \cdots & \lambda^{-M+2} \\ 0 & 0 & 0 & \lambda^{-1} & \cdots & \lambda^{-M+3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda^{-1} \end{bmatrix}.$$

Thus, if $|\lambda| = 1$, then $\left\|(\lambda - C_M)^{-1}\right\| \geq \sqrt{M}$. Also, if $|\lambda| > 1$, then $\left\|(\lambda - C_M)^{-1}\right\| \leq \sum_{j=0}^{M-1} |\lambda|^{-1-j} \leq 1/(|\lambda| - 1)$. In particular, if $|\lambda| \geq 2$, then $\left\|(\lambda - C_M)^{-1}\right\| \leq 1$.

Also note that

$$e^{tC_M} = \begin{bmatrix} 1 & t & t^2/2 & t^3/6 & \cdots & t^{M-1}/(M-1)! \\ 0 & 1 & t & t^2/2 & \cdots & t^{M-2}/(M-2)! \\ 0 & 0 & 1 & t & \cdots & t^{M-3}/(M-3)! \\ 0 & 0 & 0 & 1 & \cdots & t^{M-4}/(M-4)! \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Thus we see that e^{tC_M} is a positive operator. Clearly $\left\|e^{tC_M}\right\| \leq e^{t\|C_M\|} \leq e^t$.

Consider the positive semigroup acting on $L_2([0, 2\pi])$ by

$$e^{tA_M} f(x) = (e^{4t} - 1) \int_0^{2\pi} f(x)\, \frac{dx}{2\pi} + f(x + Mt),$$

so that its generator is the closure of

$$A_M f(x) = 4 \int_0^{2\pi} f(x)\, \frac{dx}{2\pi} + M \frac{d}{dx} f(x).$$

Note that $\left\|e^{tA_M}\right\| \le e^{4t}$.

Now consider the positive semigroup $e^{tB_M} = e^{tA_M} \otimes e^{tC_M}$ acting on

$$X_M = L_2([0,2\pi]) \otimes \ell_2^M = L_2([0,2\pi] \times \{1,2,\dots,M\}).$$

We see that this semigroup is generated by $B_M = A_M \otimes I + I \otimes C_M$. Also, $\left\|e^{tB_M}\right\| \le e^{5t}$.

Consider a typical element of X_M given by $f(x) = \sum_{n=-\infty}^{\infty} v_n e^{inx}$, where $v_n \in \ell_2^M$ and $\|f\|_{X_M}^2 = 2\pi \sum_{n=-\infty}^{\infty} \|v_n\|_2^2$. If $\lambda \ne 4$ and $\lambda \notin M\mathbb{Z} \setminus \{0\}$, then $\lambda \notin \sigma(B_M)$ and

$$(\lambda - B_M)^{-1} f(x) = (\lambda - 4 - C_M)^{-1} v_0 + \sum_{n \ne 0} (\lambda - inM - C_M)^{-1} v_n e^{inx}.$$

Thus

$$\left\|(\lambda - B_M)^{-1}\right\| = \max\left\{ \left\|(\lambda - 4 - C_M)^{-1}\right\|, \sup_{n\ne 0} \left\|(\lambda - inM - C_M)^{-1}\right\| \right\}.$$

In particular, if $\mathrm{Re}(\lambda) = 1$ and $|\lambda| \le M - 2$, then $\left\|(\lambda - B_M)^{-1}\right\| \le 1$, whereas if $\lambda = 1 + iM$, then $\left\|(\lambda - B_M)^{-1}\right\| \ge \sqrt{M}$.

Now consider the semigroup $e^{tD} = \bigoplus_{M=1}^{\infty} e^{tB_M}$ acting on

$$\bigoplus_{M=1}^{\infty} X_M = L_2\left(\bigvee_{M=1}^{\infty} \left([0,2\pi] \times \{1,2,\dots,M\}\right) \right).$$

Note that e^{tD} really is a strongly continuous semigroup, with $\left\|e^{tD}\right\| \le e^{5t}$. The generator D is the closure of $\bigoplus_{M=1}^{\infty} B_M$, and hence its resolvent set consists of those λ such that

$$\left\|(\lambda - D)^{-1}\right\| = \sup_{M \ge 1} \left\|(\lambda - B_M)^{-1}\right\| < \infty,$$

that is, $\sigma(D) \subseteq \{z : |z - 4| \le 1\} \cup i\mathbb{Z} \setminus \{0\}$. In particular, if $\mathrm{Re}(\lambda) = 1$, then $\lambda \notin \sigma(D)$. However, $\sup_{\lambda \in 1 + i\mathbb{Z}} \left\|(\lambda - D)^{-1}\right\| = \infty$, and hence, by Gerhard's Theorem (see [N], p. 95), $e^{2\pi} \in \sigma(e^{2\pi D})$.

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