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EXTREMAL PROBLEMS IN MINKOWSKI SPACE RELATED TO MINIMAL NETWORKS

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ABSTRACT. We solve the following problem of Z. Füredi, J. C. Lagarias and F. Morgan (1991): Is there an upper-bound polynomial in n for the largest cardinality of a set S of unit vectors in an n-dimensional Minkowski space (or Banach space) such that the sum of any subset has norm less than 1? We prove that $|S| \leq 2n$ and that equality holds iff the space is linearly isometric to ℓ^n_∞, the space with an n-cube as unit ball. We also remark on similar questions they raised that arose out of the study of singularities in length-minimizing networks in Minkowski spaces.

1. INTRODUCTION

In [LM] Lawlor and Morgan derived a geometrical description for the singularities (Steiner points) of a length-minimizing network connecting a finite set of points in a smooth Minkowski space (finite-dimensional Banach space). In Euclidean space the geometrical description is equivalent to the classical result that at a singularity three line segments meet at 120° angles. See also [BG], [M] and [CR] for a discussion of length-minimizing networks and their history. The geometrical description of Lawlor and Morgan leads to extremal problems of a combinatorial type in strictly convex Minkowski spaces. Such problems are considered in [FLM]. In this note we briefly remark on some of these problems and solve one of the open problems stated in [FLM] (see Theorem 3).

2. PRELIMINARIES

We denote the real numbers by $\mathbb{R}$ and the real vector space of n-tuples of real numbers by $\mathbb{R}^n$. The coordinates of a vector $\mathbf{x} \in \mathbb{R}^n$ will be denoted by $\mathbf{x} = (\mathbf{x}(1), \mathbf{x}(2), \ldots, \mathbf{x}(n))$. The standard basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ will be used, where $\mathbf{e}_i$ is the vector for which $\mathbf{e}_i(i) = 1$ and $\mathbf{e}_i(j) = 0$ for $i \neq j$. A *Minkowski space* (or finite-dimensional Banach space) $(\mathbb{R}^n, \Phi)$ is $\mathbb{R}^n$ endowed with a norm Φ. A Minkowski space is *strictly convex* if $\Phi(\mathbf{x}) = \Phi(\mathbf{y}) = 1, \mathbf{x} \neq \mathbf{y}$ implies $\Phi(\mathbf{x} + \mathbf{y}) < 2$.

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We denote by ℓ_p^n the n-dimensional Minkowski space with norm

$$\Phi_p(\mathbf{x}) = \left(\sum_{i=1}^n |\mathbf{x}(i)|^p\right)^{1/p}$$

for $p \geq 1$, and by ℓ_∞^n the space with norm

$$\Phi_\infty(\mathbf{x}) = \max_{1\leq i\leq n} |\mathbf{x}(i)|.$$

We now state Auerbach's Lemma which relates the spaces ℓ_1^n and ℓ_∞^n to an arbitrary Minkowski space in n dimensions. A proof may be found in [Pi, page 29].

Auerbach's Lemma. *For any Minkowski space* $(\mathbb{R}^n, \Phi)$ *there exists a linear isomorphism* $T : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $\Phi_\infty(\mathbf{x}) \leq \Phi(T\mathbf{x}) \leq \Phi_1(\mathbf{x})$, *i.e.*

$$\max_{1\leq i\leq n} |\mathbf{x}(i)| \leq \Phi(T\mathbf{x}) \leq \sum_{i=1}^n |\mathbf{x}(i)|. \tag{1}$$

We denote the n-dimensional Lebesgue measure (or *volume*) of measurable $V \subseteq \mathbb{R}^n$ by $\mathrm{vol}(V)$. If $U, V \subseteq \mathbb{R}^n$, then we define $U + V = \{\mathbf{u}+\mathbf{v} \mid \mathbf{u} \in U, \mathbf{v} \in V\}$. The Brunn-Minkowski inequality relates the volumes of compact U and V to that of $U + V$. A proof may be found in [BZ].

Brunn-Minkowski inequality. *If* $U, V \subseteq \mathbb{R}^n$ *are compact, then*

$$(\mathrm{vol}(U + V))^{1/n} \geq (\mathrm{vol}(U))^{1/n} + (\mathrm{vol}(V))^{1/n}.$$

3. Extremal problems

From now on S will denote a finite set of unit vectors in a Minkowski space. In [FLM] the following type of extremal problems is considered: Find the largest cardinality of S satisfying a selection of the following conditions:

(A)	$\Phi(\sum_{\mathbf{x}\in J} \mathbf{x}) \leq 1$ for all $J \subseteq S$	(the *strong collapsing condition*),
(A')	$\Phi(\mathbf{x}+\mathbf{y}) \leq 1$ for all $\mathbf{x}, \mathbf{y} \in S, \mathbf{x} \neq \mathbf{y}$	(the *weak collapsing condition*),
(B)	$\sum_{\mathbf{x}\in S} \mathbf{x} = \mathbf{0}$	(the *strong balancing condition*),
(B')	$\mathbf{0}$ is in the relative interior of the convex hull of S	(the *weak balancing condition*).

See [LM] and [FLM] for the connection between these conditions and minimal networks. In [FLM] it is proved that (A') and (B') together give an upper bound $|S| \leq 2n$ for an arbitrary Minkowski space, and $|S| \leq n + 1$ for strictly convex Minkowski spaces. In [LM] it is proved that there exist a strictly convex norm on $\mathbb{R}^n$ and a subset S of $n+1$ unit vectors satisfying (A) and (B). $|S| = 2n$ is attained in, for example, ℓ_∞^n with $S = \{\pm\mathbf{e}_i \mid 1 \leq i \leq n\}$, in which case even the strong conditions (A) and (B) hold. However, there are other Minkowski spaces where equality is also attained (see Theorem 1). This is to be contrasted with Theorem 3, where we show that the extreme case for S satisfying (A) and (B) can only be attained for ℓ_∞^n.

Theorem 1. *For infinitely many* $n \geq 1$ *there exists a set of unit vectors* $S = \{\mathbf{x}_1, \dots, \mathbf{x}_{2n}\} \subseteq \ell_1^n$ *satisfying* (A') *and the strong balancing condition* (B). *In particular, such a set exists if a Hadamard matrix of order* n *exists.*

Proof. We recall that an $n\times n$ Hadamard matrix H consists of (± 1)-entries such that $HH^t = nI$, and such matrices exist for infinitely many n (see [vLW, Chapter 18]). We let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be the column vectors of H, and set $\mathbf{x}_i = \frac{1}{n}\mathbf{v}_i$ for $i = 1, \dots, n$. Then $S := \{\pm\mathbf{x}_i \mid 1 \le i \le n\}$ is a set of $2n$ unit vectors. Since the column vectors of H are orthogonal, $\langle \mathbf{v}_i, \mathbf{v}_j\rangle = 0$ for $i \ne j$, implying that $\Phi_1(\mathbf{x}_i + \mathbf{x}_j) = 1$ and $\Phi_1(\mathbf{x}_i - \mathbf{x}_j) = 1$ for all $i \ne j$. It follows that S satisfies (A') and (B). □

The question now is what happens if there is no balancing condition present. In [FLM] an upper bound of $|S| < 3^n$ is derived from the weak collapsing condition (A') alone using a volume argument. Using the Brunn-Minkowski inequality we obtain a sharper bound (Theorem 2). In [FLM] a strictly convex norm and a set S of unit vectors with $|S| \ge (1.02)^n$ satisfying (A') are constructed for all sufficiently large n. It would be interesting to find the greatest lower bound of the α's for which $|S| \le \alpha^n$ for any set S of unit vectors in an arbitrary Minkowski space satisfying (A'), and sufficiently large n.

Theorem 2. *If a set S of unit vectors in $\mathbb{R}^n$ satisfies (A'), then $|S| < 2^{n+1}$.*

Proof. We denote the closed unit ball with centre $\mathbf{x}$ and radius r by $B(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^n \mid \Phi(\mathbf{x} - \mathbf{y}) \le r\}$, and the volume of a ball of unit radius by β. For distinct $\mathbf{x}, \mathbf{y} \in S$ we obtain from the triangle inequality that $\Phi(\mathbf{x} - \mathbf{y}) \ge 1$. Let $k = |S|$. We partition S into two sets S_1 and S_2 of sizes $\lfloor k/2\rfloor$ and $\lceil k/2\rceil$, respectively. Let $V_i = B(\mathbf{0}, \frac{1}{2}) \cup \bigcup_{\mathbf{x}\in S_i} B(\mathbf{x}, \frac{1}{2})$ for $i = 1, 2$. Clearly, each V_i consists of closed balls with disjoint interiors, and therefore, $\mathrm{vol}(V_1) = \beta(\lfloor k/2\rfloor + 1)2^{-n}$ and $\mathrm{vol}(V_2) = \beta(\lceil k/2\rceil + 1)2^{-n}$. Using (A') we obtain $V_1 + V_2 \subseteq B(\mathbf{0}, 2)$ and $\mathrm{vol}(V_1 + V_2) \le 2^n\beta$. By the Brunn-Minkowski inequality we now have

$$2\beta^{1/n} \ge \tfrac{1}{2}\beta^{1/n}(\lfloor k/2\rfloor + 1)^{1/n} + \tfrac{1}{2}\beta(\lceil k/2\rceil + 1)^{1/n} > \beta^{1/n}(k/2)^{1/n}$$

and $|S| < 2^{n+1}$. □

From the above proof we actually find that if $\Phi(\mathbf{x}) = \Phi(\mathbf{y}) = 1$ and $\Phi(\mathbf{x}+\mathbf{y}) \le 1$ imply $\Phi(\mathbf{x} - \mathbf{y}) \ge r > 1$, then $|S| \le 2(1 + 1/r)^n + 1$ for S satisfying (A'). Such is the case for ℓ_p^n: It follows from the Clarkson inequality [C] for $p \ge 2$, and the Hanner inequality [H] for $1 < p < 2$, that r may be taken to be $3^{1/p}$ for $p \ge 2$, and $(2^p - 1)^{1/p}$ for $1 < p < 2$.

For ℓ_1^n an upper bound $|S| \le 2^n$ holds: If the coordinates of two unit vectors $\mathbf{x}$ and $\mathbf{y}$ have the same sequence of signs, i.e. $\mathrm{sgn}(\mathbf{x}(i)) = \mathrm{sgn}(\mathbf{y}(i))$ for all $i = 1, \dots, n$, then $\Phi_1(\mathbf{x}+\mathbf{y}) = 2$, contradicting (A'). In the Euclidean case ℓ_2^n we of course have $|S| \le 3$, independent of n. For ℓ_∞^n the sharp upper bound $|S| \le 2n$ holds: If $|S| \ge 2n + 1$, then by the pigeon-hole principle there are three vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in S$ and an $i \in \{1, \dots, n\}$ such that $|\mathbf{x}(i)| = |\mathbf{y}(i)| = |\mathbf{z}(i)| = 1$. Some two of these vectors will have the same sign in the ith coordinate, and their sum will then have a norm of 2.

In [FLM, Problem 3.7] the question is asked whether the strong collapsing condition (A) on its own gives an upper bound for $|S|$ that is polynomial in n. A linear upper bound may be derived by the same technique as in Theorem 2. We partition the elements of S except for at most 2 into subsets $S_1, \dots, S_k$ of size 3, where $k = \lfloor |S|/3\rfloor$. For $i = 1, \dots, k$ let

$$V_i = \bigcup_{\mathbf{x}\in S_i} B(\mathbf{x}, \tfrac{1}{2}) \cup \bigcup_{\mathbf{x},\mathbf{y}\in S_i, \mathbf{x}\ne\mathbf{y}} B(\mathbf{x}+\mathbf{y}, \tfrac{1}{2}).$$

From (A) it follows that each V_i consists of 6 balls with disjoint interiors and $V_1 + \cdots + V_k \subseteq B(\mathbf{0}, \frac{1}{2}k+1)$. By the Brunn-Minkowski inequality we obtain $\frac{1}{2}k+1 \geq \frac{1}{2}6^{1/n}k$ and $k \leq 2/(6^{1/n}-1)$. Therefore, $|S| \leq 6/(6^{1/n}-1)+2 < (6/\ln 6)n$, after some calculus. This bound is not sharp, however. In the following theorem we derive the sharp upper bound $|S| \leq 2n$.

Theorem 3. *Let S be a finite set of unit vectors in a Minkowski space $(\mathbb{R}^n, \Phi)$ satisfying the collapsing condition (A). Then $|S| \leq 2n$, and equality holds iff $(\mathbb{R}^n, \Phi)$ is linearly isometric to ℓ_∞^n, with S corresponding to the set $\{\pm \mathbf{e}_i \mid 1 \leq i \leq n\}$ under any isometry.*

Proof. By Auerbach's Lemma we may assume (after applying a linear isomorphism of $\mathbb{R}^n$) that for any vector $\mathbf{x} \in \mathbb{R}^n$ the inequalities (1) hold, with T now the identity. Choose m distinct vectors $\mathbf{x}_1, \dots, \mathbf{x}_m$ from S. By (1) we have

$$\sum_{i=1}^{n} |\mathbf{x}_j(i)| \geq 1 \text{ for all } j = 1, \dots, m. \tag{2}$$

Suppose that for some coordinate $i \in \{1, \dots, n\}$ we have

$$\sum_{\substack{j=1 \\ \mathbf{x}_j(i) \geq 0}}^{n} \mathbf{x}_j(i) > 1.$$

Then

$$\Phi\Big(\sum_{\substack{j=1 \\ \mathbf{x}_j(i) \geq 0}}^{m} \mathbf{x}_j\Big) \geq \sum_{\substack{j=1 \\ \mathbf{x}_j(i) \geq 0}}^{m} \mathbf{x}_j(i)$$

by (1), contradicting (A). Therefore,

$$\sum_{\substack{j=1 \\ \mathbf{x}_j(i) \geq 0}}^{m} \mathbf{x}_j(i) \leq 1 \text{ for all } i = 1, \dots, n, \tag{3}$$

and similarly,

$$\sum_{\substack{j=1 \\ \mathbf{x}_j(i) \leq 0}}^{m} -\mathbf{x}_j(i) \leq 1 \text{ for all } i = 1, \dots, n. \tag{4}$$

From (3) and (4) it follows that $\sum_{j=1}^{m} |\mathbf{x}_j(i)| \leq 2$ for all $i = 1, \dots, n$, and from (2) we have

$$m \leq \sum_{j=1}^{m} \sum_{i=1}^{n} |\mathbf{x}_j(i)| \leq 2n \tag{5}$$

and $|S| \leq 2n$.

If $|S| = 2n$ for some set of unit vectors $S = \{\mathbf{x}_1, \dots, \mathbf{x}_{2n}\}$ satisfying (A), then equality must hold in (5), (3) and (4). Therefore, $\sum_{j=1}^{2n} \mathbf{x}_j = 0$, showing that in the extreme case the strong balancing condition (B) must be satisfied. We now show that conditions (A) and (B) together with the assumption $|S| = 2n$ imply that $(\mathbb{R}^n, \Phi)$ is linearly isometric to ℓ_∞^n, and S corresponds to $\{\pm \mathbf{e}_i \mid 1 \leq i \leq n\}$, as claimed in [FLM].

We recall Theorem 3.1 of [FLM]:

If $(\mathbb{R}^n, \Phi)$ is a Minkowski space and S is a set of unit vectors satisfying (A') and (B'), then $|S| \leq 2n$, and if equality holds, then S corresponds to $\{\pm \mathbf{e}_i \mid 1 \leq i \leq n\}$ under some linear isomorphism.

We therefore have $S = \{\pm \mathbf{x}_i \mid i = 1 \leq i \leq n\}$, where the $\mathbf{x}_i$'s are linearly independent. We first show that if $(\mathbb{R}^n, \Phi) = \ell_\infty^n$, then $S = \{\pm \mathbf{e}_i \mid 1 \leq i \leq n\}$ must hold. For $i = 1, \ldots, n$ choose $j_i \in \{1, \ldots, n\}$ such that $|\mathbf{x}_i(j_i)| = 1$. After renaming, we may assume $\mathbf{x}_i(j_i) = 1$. The j_i's must be distinct, otherwise (A) is contradicted. We may therefore rename the $\mathbf{x}_i$'s to obtain $\mathbf{x}_i(i) = 1$ for $i = 1, \ldots, n$. If we have $\mathbf{x}_i(j) \neq 0$ for some $i \neq j$, then either $\Phi_\infty(\mathbf{x}_i + \mathbf{x}_j) > 1$ or $\Phi_\infty(-\mathbf{x}_i + \mathbf{x}_j) > 1$, contradicting (A). Therefore, $\mathbf{x}_i(j) = 0$ for all $i \neq j$, and we have $S = \{\pm \mathbf{e}_i \mid i = 1, \ldots, n\}$.

To show that in fact $(\mathbb{R}^n, \Phi)$ is linearly isometric to ℓ_∞^n, we use the following theorem of Petty [Pe] (see also [FLM, Theorem 2.1]):

If T is a subset of a Minkowski space $(\mathbb{R}^n, \Phi)$ such that $\Phi(\mathbf{x} - \mathbf{y}) = 1$ for all $\mathbf{x}, \mathbf{y} \in T, \mathbf{x} \neq \mathbf{y}$, then $|T| \leq 2^n$, with equality iff $(\mathbb{R}^n, \Phi)$ is linearly isometric to ℓ_∞^n.

We will apply this theorem to the set $T = \{\sum_{i \in A} \mathbf{x}_i \mid A \subseteq \{1, \ldots, n\}\}$. Obviously $|T| = 2^n$. We now show that

$$\Phi(\sum_{i \in A} \mathbf{x}_i - \sum_{i \in B} \mathbf{x}_i) = 1 \text{ for all } A, B \subseteq \{1, \ldots, n\},\ A \neq B, \tag{6}$$

thus completing the proof. Firstly, we have

$$\Phi(\sum_{i \in A} \mathbf{x}_i - \sum_{i \in B} \mathbf{x}_i) = \Phi(\sum_{i \in A \setminus B} \mathbf{x}_i + \sum_{i \in B \setminus A} -\mathbf{x}_i) \leq 1$$

by (A). Secondly, $A \setminus B \neq \emptyset$ or $B \setminus A \neq \emptyset$, since $A \neq B$. We assume without loss that $A \setminus B \neq \emptyset$ and choose $j \in A \setminus B$. Then

$$\begin{aligned} 2 = \Phi(2\mathbf{x}_j) &\leq \Phi(\sum_{i \in A} \mathbf{x}_i - \sum_{i \in B} \mathbf{x}_i) + \Phi(\sum_{i \in B \cup \{j\}} \mathbf{x}_i - \sum_{i \in A \setminus \{j\}} \mathbf{x}_i) \\ &\leq \Phi(\sum_{i \in A} \mathbf{x}_i - \sum_{i \in B} \mathbf{x}_i) + 1, \end{aligned}$$

showing that (6) holds. □

For strictly convex norms the bound in the above theorem should perhaps be $|S| \leq n + 1$, but this seems to require a new idea.

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