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ON RELATIVE HAUSDORFF MEASURES OF NONCOMPACTNESS AND RELATIVE CHEBYSHEV RADII IN BANACH SPACES

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ABSTRACT. In this paper we prove some formulae and evaluations on relative Hausdorff measures of noncompactness and relative Chebyshev radii in various Banach spaces. We generalize the Lifschitz constant $\kappa(X)$ and introduce a function $\tilde{\kappa}_X(\cdot)$.

1. INTRODUCTION

In this paper X will always denote a real Banach space and A will be a bounded subset of X. The classical Hausdorff measure of noncompactness $\chi(A)$ is defined as the infimum of numbers $\varepsilon > 0$ such that A can be covered with a finite number of balls of radii smaller than ε. The absolute Chebyshev radius $r(A)$ is defined as the infimum of numbers $\varepsilon > 0$ such that A can be covered with a ball of a radius ε. Thus we have

$$r(A) = \inf_{y \in X} \sup_{x \in A} \|x - y\|.$$

The concept of relative Chebyshev centers and radii is a natural generalization of the notion of the absolute Chebyshev center and radius. In particular, relative Chebyshev centers have been well studied in recent years (see for instance [13]). For a given set $\emptyset \neq G \subset X$ the relative radius $r_G(A)$ is given by

$$r_G(A) = \inf_{y \in G} \sup_{x \in A} \|x - y\|.$$

Similarly we can define the relative Hausdorff measure of noncompactness $\chi_G(A)$ as the infimum of those $r > 0$ such that A can be covered with a finite number of balls with centers in G of radii smaller than r. If $G = A$ we have the so-called inner Hausdorff measure of noncompactness. If G is a linear subspace of X and $A \subset G$, we obtain the (classical) Hausdorff measure of noncompactness in a subspace G [16].

Section 2 contains formulae on $\chi_G(A)$ and $r_G(A)$ in terms of the Hausdorff distance.

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In section 3 we generalize the Lifschitz constant $\kappa(X)$ and give evaluations of $\chi_G(A)$ and $r_G(A)$ in any Banach spaces.

In section 4 a function $\tilde{\kappa}_X(\cdot)$ is defined. With the help of this function we give some stronger evaluations of $\chi_G(A)$ and $r_G(A)$ than those given in section 3.

Section 5 presents some applications of previous ideas to Hilbert spaces, L^p spaces and some spaces with the norm "supremum". We generalize the formula proved by Smith and Ward in [15].

2. General remarks

Let $B(x,r)$ denote the closed ball centered at $x \in X$ with radius $r > 0$, and let $\mathrm{dist}(x,A)$ denote the distance to a point x from $A \subset X$. We shall also use the notation:

$$B(A,r) = \{x \in X : \mathrm{dist}(x,A) \leq r\},$$

$$\mathrm{dist}(A,G) = \inf_{x\in A} \inf_{y\in G} \|x-y\|, \qquad A, G \subset X.$$

Denote by $\mathcal{M}$ the family of all nonempty bounded subsets of X. For $A, G \in \mathcal{M}$ put

$$d(A,G) = \inf\{r > 0 : A \subset B(G,r)\},$$

$$D(A,G) = \max\{d(A,G), d(G,A)\}$$

and call them the nonsymmetric and symmetric Hausdorff distance between A and G, respectively. Sometimes we shall use the symbol $d(A,G)$ with unbounded G. It is well known that D is a metric defined on the family of all bounded and closed subsets of X. We shall also use the following symbols:

$\mathcal{N}$–the family of all relatively compact and nonempty subsets of X,

$\mathcal{N}^0$–the family of all nonempty finite subsets of X,

$\mathcal{N}^s$–the family of all subsets of X consisting of exactly one element.

If $\mathcal{Z}$ is a family of subsets of X, then we shall write

$$d(A,\mathcal{Z}) = \inf_{G\in\mathcal{Z}} d(A,G) \quad \text{and} \quad D(A,\mathcal{Z}) = \inf_{G\in\mathcal{Z}} D(A,G).$$

Using this notation it is easily seen that

$$\chi(A) = D(A,\mathcal{N}) = d(A,\mathcal{N}) = D(A,\mathcal{N}^0) = d(A,\mathcal{N}^0)$$

and

$$r(A) = D(A,\mathcal{N}^s) = d(A,\mathcal{N}^s).$$

Proposition 2.1. *Let $\emptyset \neq G \subset X$. Then*

$$\chi_G(A) = \inf_{F\in\mathcal{N}}[d(A,F) + d(F,G)] = \inf_{F\in\mathcal{N}^0}[d(A,F) + d(F,G)],$$

$$r_G(A) = \inf_{F\in\mathcal{N}^s}[d(A,F) + d(F,G)].$$

Proof. The equality $\inf_{F\in\mathcal{N}}[d(A,F)+d(F,G)] = \inf_{F\in\mathcal{N}^0}[d(A,F)+d(F,G)]$ follows from the fact that for every $\varepsilon > 0$ and $F \in \mathcal{N}$ there exists $F_0 \in \mathcal{N}^0$ such that $d(F,F_0) < \varepsilon$ and $d(F_0,F) < \varepsilon$. Therefore it is sufficient to prove that $\chi_G(A) = \inf_{F\in\mathcal{N}^0}[d(A,F) + d(F,G)]$. Fix $\varepsilon > 0$ and $F \in \mathcal{N}^0$, and choose $x \in A$. By the definition of $d(A,F)$ and $d(F,G)$ there exist $f \in F$ and $g \in G$ such that $\|x-f\| \le d(A,F)+\frac{\varepsilon}{2}$ and $\|f-g\| \le d(F,G)+\frac{\varepsilon}{2}$. Hence

$$\|x-g\| \le \|x-f\|+\|f-g\| \le d(A,F)+d(F,G)+\varepsilon$$

and so

$$\chi_G(A) \le d(A,F)+d(F,G)$$

for every $F \in \mathcal{N}^0$. This implies

$$\chi_G(A) \le \inf_{F\in\mathcal{N}^0}[d(A,F)+d(F,G)].$$

We have

$$\begin{aligned}\chi_G(A) &\le \inf_{F\in\mathcal{N}^0}[d(A,F)+d(F,G)] \le \inf_{\substack{F\in\mathcal{N}^0\\ F\subset G}}[d(A,F)+d(F,G)]\\ &= \inf_{\substack{F\in\mathcal{N}^0\\ F\subset G}} d(A,F) = \chi_G(A),\end{aligned}$$

which completes the proof. Similar considerations apply to $r_G(A)$. □

Write

$$D_1(A,B) = d(A,B)+d(B,A).$$

Notice that $D_1(A,B)$ is a metric defined on the family of all bounded and closed subsets of X. From Proposition 2.1 we obtain

Proposition 2.2. *Let $A \subset X$ be a bounded, nonempty set. Then*

$$\begin{aligned}\chi_A(A) &= D_1(A,\mathcal{N}) = D_1(A,\mathcal{N}^0),\\ r_A(A) &= D_1(A,\mathcal{N}^s). \qquad \square\end{aligned} \tag{1}$$

3. Evaluations on $\chi_G(A)$ and $r_G(A)$ with the use of the function $\kappa_X(\cdot)$

For $\varepsilon \ge 0$ write

$$\mathcal{H}^\varepsilon(A) = \{F \in \mathcal{N}^0 : A \subset B(F,(1+\varepsilon)\chi(A))\},$$

$$E^\varepsilon(A) = \{y \in X : A \subset B(y,(1+\varepsilon)r(A))\}.$$

Note that for $\varepsilon > 0$ $\mathcal{H}^\varepsilon(A) \ne \emptyset$ and $E^\varepsilon(A) \ne \emptyset$ for every bounded $A \subset X$. Proposition 2.1 now gives

Proposition 3.1. *Let $\emptyset \ne G \subset X$. Then*

$$\chi_G(A) \le \chi(A) + \lim_{\varepsilon\to 0^+} d(\mathcal{H}^\varepsilon(A),G),$$

$$r_G(A) \le r(A) + \lim_{\varepsilon\to 0^+} \operatorname{dist}(E^\varepsilon(A),G). \qquad \square$$

To find the converse evaluations on $\chi_G(A)$ and $r_G(A)$ let us recall the definition of the Lifschitz constant $\kappa(X)$ of a Banach space X :

$$\kappa(X) = \sup\left\{k>0 : \bigvee_{0<\mu,\alpha<1}\ \bigwedge_{x,y\in X}\ \bigwedge_{r>0} \|x-y\| \ge (1-\mu)r \right.$$
$$\left. \Rightarrow \bigvee_{z\in X} B(x,(1+\mu)r)\cap B(y,k(1+\mu)r) \subset B(z,\alpha r)\right\}.$$

This constant plays an important role in fixed point theorems for uniformly Lipschitzian mappings. We will need to generalize it.

Definition 3.2. Let X be a Banach space. $\kappa_X(\cdot)$ is a function defined on $(0,+\infty)$ by

$$\kappa_X(d) = \sup \Bigg\{ k > 0 : \bigvee_{0<\mu,\alpha<1} \bigwedge_{x,y\in X} \bigwedge_{r>0} \|x-y\| \geq (1-\mu)rd$$

$$\Rightarrow \bigvee_{z\in X} B(x,(1+\mu)r) \cap B(y,k(1+\mu)r) \subset B(z,\alpha r) \Bigg\}.$$

Theorem 3.3. *Let X be a Banach space, $A \subset X$ be a bounded set and $\emptyset \neq G \subset X$. Then*

$$\chi_G(A) \geq \chi(A)\kappa_X\left(\frac{d_1}{\chi(A)}\right) \quad \text{if } \chi(A) \neq 0, \tag{2}$$

$$r_G(A) \geq r(A)\kappa_X\left(\frac{d_2}{r(A)}\right) \quad \text{if } r(A) \neq 0, \tag{3}$$

where

$$d_1 = \lim_{\varepsilon\to 0^+} \sup_{F\in\mathcal{H}^\varepsilon(A)} \operatorname{dist}(F,G), \qquad d_2 = \lim_{\varepsilon\to 0^+} d(E^\varepsilon(A),G).$$

Proof. Write $\chi_G(A) = k$, $\chi(A) = r$. Assume that $\frac{k}{r} < \kappa_X(\frac{d_1}{r})$. Then we can find $0 < \mu$, $\alpha < 1$ such that for every $x, y \in X$ with $\|x-y\| \geq (1-\mu)d_1$ there exists $z \in X$ satisfying

$$B(x,(1+\mu)r) \cap B(y,k(1+\mu)) \subset B(z,\alpha r).$$

Choose $g_1, g_2, \ldots, g_n \in G$ satisfying

$$A \subset \bigcup_{i=1}^{n} B(g_i,(1+\mu)k)$$

and $F = \{x_1, x_2, \ldots, x_m\} \in \mathcal{H}^\mu(A)$ such that

$$\bigwedge_{\substack{i=1,\ldots,n\\ j=1,\ldots,m}} \|g_i - x_j\| \geq \operatorname{dist}(F,G) \geq (1-\mu)d_1.$$

Obviously $A \subset \bigcup_{j=1}^m B(x_j,(1+\mu)r)$. But

$$B(g_i,(1+\mu)k) \cap B(x_j,(1+\mu)r) \subset B(z_{ij},\alpha r)$$

for some $z_{ij} \in X$ and we obtain

$$A \subset \bigcup_{i=1}^{n} B(g_i,(1+\mu)k) \cap \bigcup_{j=1}^{m} B(x_j,(1+\mu)r)$$

$$= \bigcup_{i=1}^{n}\bigcup_{j=1}^{m} B(g_i,(1+\mu)k) \cap B(x_j,(1+\mu)r) \subset \bigcup_{i=1}^{n}\bigcup_{j=1}^{m} B(z_{ij},\alpha r),$$

which contradicts $\chi(A) = r$. Thus (2) is proved. The proof for (3) is similar. □

It is now natural to consider: When is $\kappa_X(d) > 1$? First recall that the modulus of convexity of a Banach space X is the function $\delta_X : [0,2] \to [0,1]$ defined by

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon \right\}$$

and

$$\varepsilon_0(X) = \sup\{\varepsilon \in [0,2] : \delta_X(\varepsilon) = 0\}.$$

We follow the ideas of Downing and Turett [3] (see also [8]).

Proposition 3.4. *In any Banach space* X :

$$\kappa_X(d) > 1 \Leftrightarrow \varepsilon_0(X) < d.$$

Proof. Let $\varepsilon_0(X) < d$ and let $u \in B(x,(1+\mu)r) \cap B(y,(1+\mu)kr)$, where $\mu > 0$, $k > 1$ and $\|x-y\| \ge (1-\mu)dr$. Then $\|u-x\| \le (1+\mu)r$, $\|u-y\| \le (1+\mu)rk$ and taking $z = \frac{x+y}{2}$ we obtain

$$\|u-z\| \le \left(1 - \delta_X\left(\frac{(1-\mu)d}{(1+\mu)k}\right)\right)(1+\mu)kr \le \alpha r$$

with $\alpha < 1$ if μ is sufficiently small and k is sufficiently close to 1. Hence $\kappa_X(d) > 1$ (see also [8]). To prove the converse assume that $\kappa_X(d) > 1$ and $\varepsilon_0(X) \ge d$. Fix $\varepsilon > 0$. There exist $x, y \in X$ such that $\|x\| = \|y\| = 1$, $\|x-y\| = d$ and $\|\frac{x+y}{2}\| \ge 1-\varepsilon$. For any $\mu > 0$ the segment

$$[0, x+y] \subset B(x,(1+\mu)) \cap B(y,(1+\mu)).$$

But $\|x+y\| \ge 2-2\varepsilon$ and ε is arbitrary so $\kappa_X(d) = 1$, which is a contradiction. □

Notice that for every $0 < d < \frac{4}{\varepsilon_0(X)}$ there exists exactly one number $b \ge 1$ such that $b(1-\delta_X(\frac{d}{b})) = 1$ (it follows from the fact that δ_X is an increasing, continuous function on $[0,2)$ and $\lim_{\varepsilon\to 2^-} \delta_X(\varepsilon) = 1 - \frac{\varepsilon_0(X)}{2}$ [7]). Denote by $b_X(\cdot)$ a function depending on the argument $d < \frac{4}{\varepsilon_0(X)}$ satisfying

$$b_X(d)\left(1 - \delta_X\left(\frac{d}{b_X(d)}\right)\right) = 1.$$

Similar considerations to that given above lead us to

Proposition 3.5. *In any Banach space* X :

$$\kappa_X(d) \ge b_X(d) \quad \textit{for } d < \frac{4}{\varepsilon_0(X)}. \qquad \square$$

The constant $b_X(1)$ was introduced by Goebel and Kirk in [6].

4. Evaluations on $\chi_G(A)$ and $r_G(A)$ with the use of the function $\tilde{\kappa}_X(\cdot)$

In section 3 the expressions d_1, d_2 in Theorem 3.3 are not natural. Now we find stronger and more useful evaluations on $\chi_G(A)$ and $r_G(A)$. We start with the following definition:

Definition 4.1. Let X be a Banach space. $\tilde{\kappa}_X(\cdot)$ is a function defined on $(0,+\infty)$ by

$$\tilde{\kappa}_X(d) = \sup\left\{ k>0 : \bigvee_{0<\alpha<1} \bigwedge_{x,y\in X} \bigwedge_{r>0} \bigvee_{z\in X} \|z-y\| \le \alpha d r \wedge B(x,r) \cap B(y,kr) \subset B(z,r) \right\}.$$

Theorem 4.2. *Let X be a Banach space, $A \subset X$ be a bounded set and $\emptyset \neq G \subset X$. Then*

$$\chi_G(A) \ge \chi(A)\tilde{\kappa}_X\left(\frac{d_\chi}{\chi(A)}\right) \quad \textit{if } \chi(A)\neq 0, \tag{4}$$

$$r_G(A) \ge r(A)\tilde{\kappa}_X\left(\frac{d_r}{r(A)}\right) \quad \textit{if } r(A)\neq 0, \tag{5}$$

where

$$d_\chi = \lim_{\varepsilon\to 0^+} d(\mathcal{H}^\varepsilon(A),G), \qquad d_r = \lim_{\varepsilon\to 0^+} \operatorname{dist}(E^\varepsilon(A),G).$$

Proof. Write $\chi_G(A)=k$, $\chi(A)=r$. Assume that $\frac{k}{r} < \tilde{\kappa}_X(\frac{d_\chi}{\chi(A)})$. Fix $\varepsilon>0$ and take $0<\delta\le\varepsilon$ such that $d(\mathcal{H}^\delta(A),G)\ge d_\chi(1-\varepsilon)$. Then we can find $\alpha<1$ such that for every $x,y\in X$ there exists $z\in X$ satisfying

$$\|z-y\| \le \alpha d_\chi(1+\delta) \le \alpha d_\chi(1+\varepsilon)$$

and

$$B(x,(1+\delta)r)\cap B(y,(1+\delta)k) \subset B(z,(1+\delta)r).$$

Choose $g_1,g_2,\dots,g_n\in G$ such that

$$A \subset \bigcup_{i=1}^n B(g_i,(1+\delta)k)$$

and $F=\{x_1,x_2,\dots,x_m\}\in\mathcal{H}^\delta(A)$. Then

$$B(g_i,(1+\delta)k)\cap B(x_j,(1+\delta)r) \subset B(z_{ij},(1+\delta)r)$$

for some $z_{ij}\in X$ satisfying

$$\|z_{ij}-g_i\| \le \alpha d_\chi(1+\varepsilon), \qquad i=1,\dots,n,\ j=1,\dots,m. \tag{6}$$

Hence

$$\begin{aligned} A &\subset \bigcup_{i=1}^n B(g_i,(1+\delta)k) \cap \bigcup_{j=1}^m B(x_j,(1+\delta)r) \\ &= \bigcup_{i=1}^m \bigcup_{j=1}^m B(g_i,(1+\delta)k)\cap B(x_j,(1+\delta)r) \subset \bigcup_{i=1}^n \bigcup_{j=1}^m B(z_{ij},(1+\delta)r) \end{aligned}$$

and $\{z_{ij}\}\in\mathcal{H}^\delta(A)$. From the definition of $d(\mathcal{H}^\delta(A),G)$ there exist i_0,j_0 such that

$$\|z_{i_0j_0}-g_{i_0}\| \ge d(\mathcal{H}^\delta(A),G) \ge d_\chi(1-\varepsilon),$$

which contradicts (6) if ε is sufficiently small. Thus (4) is proved. The proof for (5) is similar. ∎

Let $l_{x,y}$ denote the line $\{\alpha x+\beta y : \alpha,\beta \in \mathcal{R}, \alpha+\beta=1\}$ containing $x, y \in X$, and let $[x, y]$ denote the segment $\{\alpha x + \beta y : \alpha, \beta \geq 0, \alpha + \beta = 1\}$. In the next section we will need the following

Lemma 4.3. *In any Banach space X and for every $d > 0$ we have*

$$\tilde{\kappa}_X(d) \geq \tilde{\kappa}_X^{lin}(d),$$

where

$$\tilde{\kappa}_X^{lin}(d) = \sup\Bigg\{ k > 0 : \bigvee_{0<\alpha<1} \bigwedge_{x,y\in X} \bigwedge_{r>0} \|x-y\| \leq rd \Rightarrow \bigvee_{z\in l_{x,y}} \|z-y\| \leq \alpha rd \wedge B(x,r)\cap B(y,kr) \subset B(z,r) \Bigg\}.$$

Proof. Obviously $d-1 \leq \tilde{\kappa}_X^{lin}(d) \leq d+1$. It is sufficient to consider $\tilde{\kappa}_X^{lin}(d) > d-1$. Assume that $\tilde{\kappa}_X(d) < \tilde{\kappa}_X^{lin}(d)$ for some $d > 0$. Then there exist $\alpha < 1, x, y \in X$, $r > 0$ satisfying $\|x-y\| > dr$ and $x_1, z \in [x, y]$ such that $\|z-y\| \leq \alpha rd$, $\|x_1-y\| = dr$ and

$$B(x,r) \cap B(y,kr) \not\subset B(z,r) \supset B(x_1,r) \cap B(y,kr).$$

That means there exists $g \in X$ satisfying $\|g-x\| < r$, $\|g-y\| \leq kr$ and $\|g-x_1\| > r$. Let $y_1 \in l_{x,y}$ and $\|y-y_1\| = kr$. Then $\|x_1-y_1\| < r$ and we have $\|x_1-g_1\| = r$ for some $g_1 \in [y_1, g]$. Moreover $\|g_1-y\| \leq kr$ and thus $\|g_1-z\| \leq r$. But $\|g_1-x\| < r$ (since $\|x-y_1\| \leq r$ and $\|x-g\| < r$). This contradicts $\|x_1-g_1\| = r$. □

5. Applications

In this section we use the results of section 4. First observe that (in particular) in reflexive spaces $E^0(A) \neq \emptyset$ for every bounded set A (see [5], [9] for more details).

Let H be a Hilbert space. It is not difficult to verify that $\kappa_H(d) = \sqrt{1+d^2}$ and $\tilde{\kappa}_H(d) = \sqrt{1+d^2}$. Combining Proposition 3.1 and Theorem 4.2 we get

Corollary 5.1. *Let H be a Hilbert space, $A \subset H$ be a bounded set and $\emptyset \neq G \subset H$. Then*

$$\lim_{\varepsilon\to 0^+} \sqrt{[\chi(A)]^2 + [d(\mathcal{H}^\varepsilon(A), G)]^2} \leq \chi_G(A) \leq \chi(A) + \lim_{\varepsilon\to 0^+} d(\mathcal{H}^\varepsilon(A), G),$$

$$\sqrt{[r(A)]^2 + [\operatorname{dist}(E^0(A), G)]^2} \leq r_G(A) \leq r(A) + \operatorname{dist}(E^0(A), G). \qquad \square$$

Let $X = L^p$ or l^p, $p > 2$. We will adapt methods contained in [11], where it is proved that

$$\|tx + (1-t)y\|^p + t^{\frac{p}{2}}(1-t)^{\frac{p}{2}}\|x-y\|^p \leq t\|x\|^p + (1-t)\|y\|^p$$

for $x, y \in X, 0 \leq t \leq 1$. Take $\|x\| \leq r(1+\mu)$, $\|y\| \leq rk(1+\mu)$ and $\|x-y\| \geq (1-\mu)dr$. We have

$$\|tx + (1-t)y\| \leq (t(1+\mu)^p + (1-t)(1+\mu)^p k^p - t^{\frac{p}{2}}(1-t)^{\frac{p}{2}}(1-\mu)^p d^p)^{\frac{1}{p}} r.$$

If we find $t_0 \in (0,1)$ such that

$$t_0 + (1-t_0)k^p - t_0^{\frac{p}{2}}(1-t_0)^{\frac{p}{2}} d^p < 1, \tag{7}$$

then $k < \kappa_X(d)$. Moreover $\|t_0x+(1-t_0)y-y\| = t_0\|x-y\| \le t_0dr$ if $\|x-y\| \le dr$. Hence there would be also $k < \tilde{\kappa}_X^{lin}(d)$. But (7) is equivalent to

$$k < \left(1+t_0^{\frac{p}{2}}(1-t_0)^{\frac{p}{2}-1}d^p\right)^{\frac{1}{p}}.$$

Putting $t_0 = \frac{p}{2(p-1)}$ we have $k < (1+\frac{d^p}{2^{p-1}}p^{\frac{p}{2}}(p-1)^{1-p}(p-2)^{\frac{p}{2}-1})^{\frac{1}{p}}$. Hence (see also [11])

$$\kappa_X(d) \ge \left(1+\frac{d^p}{2^{p-1}}p^{\frac{p}{2}}(p-1)^{1-p}(p-2)^{\frac{p}{2}-1}\right)^{\frac{1}{p}} > \left(1+\frac{d^p}{2^{p-1}}\right)^{\frac{1}{p}}$$

and

$$\tilde{\kappa}_X^{lin}(d) \ge \left(1+\frac{d^p}{2^{p-1}}p^{\frac{p}{2}}(p-1)^{1-p}(p-2)^{\frac{p}{2}-1}\right)^{\frac{1}{p}} > \left(1+\frac{d^p}{2^{p-1}}\right)^{\frac{1}{p}}.$$

Proposition 3.1, Theorem 4.2 and Lemma 4.3 yield

Corollary 5.2. *Let $X = L^p$ or l^p, $p > 2$, $A \subset X$ be a bounded set and $\emptyset \ne G \subset X$. Then*

$$\lim_{\varepsilon\to 0^+}\left([\chi(A)]^p+\frac{[d(\mathcal{H}^\varepsilon(A),G)]^p}{2^{p-1}}p^{\frac{p}{2}}(p-1)^{1-p}(p-2)^{\frac{p}{2}-1}\right)^{\frac{1}{p}} \le \chi_G(A) \le \chi(A)+\lim_{\varepsilon\to 0^+}d(\mathcal{H}^\varepsilon(A),G),$$

$$\left([r(A)]^p+\frac{[\mathrm{dist}(E^0(A),G)]^p}{2^{p-1}}p^{\frac{p}{2}}(p-1)^{1-p}(p-2)^{\frac{p}{2}-1}\right)^{\frac{1}{p}} \le r_G(A) \le r(A)+\mathrm{dist}(E^0(A),G). \qquad \square$$

Let $X = L^p$ or l^p, $1 < p \le 2$. Lim, Xu, and Xu [12] and Smarzewski [14] proved that

$$\|tx+(1-t)y\|^2+(p-1)t(1-t)\|x-y\|^2 \le t\|x\|^2+(1-t)\|y\|^2$$

for $x, y \in X$ and $0 < t < 1$. Similar considerations to those given above give $\kappa_X(d) \ge \sqrt{1+(p-1)d^2}$ and $\tilde{\kappa}_X^{lin}(d) \ge \sqrt{1+(p-1)d^2}$. Hence we get

Corollary 5.3. *Let $X = L^p$ or l^p, $1 < p \le 2$, $A \subset X$ be a bounded set and $\emptyset \ne G \subset X$. Then*

$$\lim_{\varepsilon\to 0^+}\sqrt{[\chi(A)]^2+(p-1)[d(\mathcal{H}^\varepsilon(A),G)]^2} \le \chi_G(A) \le \chi(A)+\lim_{\varepsilon\to 0^+}d(\mathcal{H}^\varepsilon(A),G),$$

$$\sqrt{[r(A)]^2+(p-1)[\mathrm{dist}(E^0(A),G)]^2} \le r_G(A) \le r(A)+\mathrm{dist}(E^0(A),G). \qquad \square$$

Let $X = C([0,1],\mathcal{R})$ be the space of real, continuous functions defined on $[0,1]$. We have $\kappa_X(d) = \max\{1, d-1\}$ so Theorem 3.3 is not valid in this case. However we may use the function $\tilde{\kappa}_X(\cdot)$. We formulate our theorem in a more general case:

Theorem 5.4. *Let $M(T,\mathcal{R})$ be the space of bounded, real functions defined on T with the norm $\|x\| = \sup_{t\in T}|x(t)|$. Let X be a linear subspace of $M(T,\mathcal{R})$ containing the function $x(t) \equiv 1$ and closed with respect to taking maximum and minimum, i.e.:*

$$\bigwedge_{x,y\in X} z(t) = \max\{x(t),y(t)\} \in X, \qquad \bigwedge_{x,y\in X} z(t) = \min\{x(t),y(t)\} \in X.$$

If $A \subset X$ is bounded and $\emptyset \neq G \subset X$, then

$$\chi_G(A) = \chi(A) + \lim_{\varepsilon \to 0^+} d(\mathcal{H}^\varepsilon(A), G), \tag{8}$$

$$r_G(A) = r(A) + \lim_{\varepsilon \to 0^+} \operatorname{dist}(E^\varepsilon(A), G). \tag{9}$$

Proof. It is enough to show that $\tilde{\kappa}_X(d) = 1 + d$. Let $x, y \in X$, $0 < \alpha < 1$, $r > 0$, $k < 1 + d$ and $u \in B(x, r) \cap B(y, kr)$. Write:

$$z(t) = \begin{cases} y(t) + (k-1)r & \text{if } x(t) \geq y(t) + (k-1)r, \\ x(t) & \text{if } y(t) - (k-1)r < x(t) < y(t) + (k-1)r, \\ y(t) - (k-1)r & \text{if } x(t) \leq y(t) - (k-1)r. \end{cases}$$

Then $\|z - y\| \leq (k-1)r \leq \alpha d r$ if α is sufficiently close to 1. Moreover $\|u - z\| \leq r$ so $u \in B(z, r)$ and thus $\tilde{\kappa}_X(d) \geq 1 + d$. The inequality $\tilde{\kappa}_X(d) \leq 1 + d$ is obvious. □

Remark 5.5. In particular Theorem 5.4 is valid for $C([0,1], \mathcal{R}), c, c_0, l^\infty$.

Remark 5.6. The formula (9) in the case of continuous functions was first proved by Smith and Ward in [15] and then by Franchetti and Cheney in [4].

Remark 5.7. The case $G = \{x \in X : \int_{[0,1]} x \, d\mu = 0\}$ if $X = C([0,1], \mathcal{R})$, where μ is a real, normalized Borel measure on $[0,1]$, was studied in [16]:

$$\chi_G(A) = \max\{\chi(A), \omega^+(A), \omega^-(A)\},$$

where

$$\chi(A) = \frac{1}{2} \lim_{h \to 0^+} \sup_{x \in A} \sup\{|x(t) - x(s)| : |t - s| \leq h, \ t, s \in [0,1]\},$$

$$\omega^+(A) = \lim_{h \to 0^+} \sup_{x \in A} \left(\int_{[0,1]} R_h^+ x \, d\mu_1 - \int_{[0,1]} R_h^- x \, d\mu_2 \right),$$

$$\omega^-(A) = \lim_{h \to 0^+} \sup_{x \in A} \left(\int_{[0,1]} R_h^+ x \, d\mu_2 - \int_{[0,1]} R_h^- x \, d\mu_1 \right),$$

$$(R_h^+ x)(t) = \sup\{x(t + \theta h) : -1 \leq \theta \leq 1, t + \theta h \in [0,1]\},$$

$$(R_h^- x)(t) = \inf\{x(t + \theta h) : -1 \leq \theta \leq 1, t + \theta h \in [0,1]\},$$

$\mu = \mu_1 - \mu_2$ is the Jordan decomposition of μ.

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