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CARTAN INVARIANTS OF GROUP ALGEBRAS OF FINITE GROUPS

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Dedicated to Professor Takeshi Kondo on his 60th birthday

ABSTRACT. We give a result on Cartan invariants of the group algebra kG of a finite group G over an algebraically closed field k, which implies that if the Loewy length (socle length) of the projective indecomposable kG-module corresponding to the trivial kG-module is four, then k has characteristic 2. The proof is independent of the classification of finite simple groups.

0. INTRODUCTION AND NOTATION

Let kG be the group algebra of a finite group G over a field k of characteristic $p > 0$. By a kG-module we always mean a right kG-module. In this paper we discuss Cartan invariants of kG, especially those of the projective cover $P = P(k_G)$ of the trivial kG-module k_G. Let j be the Loewy length of P, that is, j is the least positive integer t such that $PJ^t = 0$ where J is the Jacobson radical of kG. It is well-known that the structure of G is completely determined provided $j \leqslant 2$ by Maschke and Wallace [8]. The structure of G with $j = 3$ has not been determined yet. There is, however, a result by Okuyama [7, Theorem 2]. He proved that Sylow 2-subgroups of G are dihedral if $j = 3$ under the condition $p = 2$. Here we investigate finite groups G satisfying $j = 4$. As a matter of fact, the condition $j = 4$ seems stronger and more mysterious than the condition $j = 3$ as seen below.

Namely, our results of this note are the following.

Theorem. *Let kG be the group algebra of a finite group G over an algebraically closed field k of characteristic $p > 0$. Assume that S is a simple kG-module which is self-dual, that is, the dual module $S^* = \mathrm{Hom}_k(S, k)$ of S is isomorphic to S itself as kG-modules. If p is odd, then there is a simple kG-module T such that T is self-dual and the Cartan invariant $c(S, T)$ with respect to S and T is odd.*

Corollary. *Let k be an arbitrary field and G a finite group. If the Loewy length of the projective indecomposable kG-module corresponding to the trivial kG-module is four, then k has characteristic 2.*

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Note that the theorem and the corollary do not depend on the classification of finite simple groups.

Remark. It should be remarked that if G is the symmetric group on 4 letters and if k has characteristic 2, then the Loewy length of the projective indecomposable kG-module corresponding to the trivial kG-module is four. The situation for p-solvable groups is treated completely in our previous paper [2].

Throughout this paper we use the following notation and terminology. By a kG-module we mean a finitely generated right kG-module. We write k_G for the trivial kG-module, and $P(k_G)$ for its projective cover. We denote by $J(kG)$ the Jacobson radical of kG. Let M be a kG-module. Then $j(M)$ denotes the Loewy length of M, namely, $j(M)$ is the least positive integer j such that $M \cdot J(kG)^j = 0$, and $\mathrm{Soc}(M)$ denotes the socle of M (see [3, I §8]). We write M^* for the dual module of M, that is, $M^* = \mathrm{Hom}_k(M, k)$ and this is also a right kG-module (see [3, I §6]). Then, M is called *self-dual* if $M \simeq M^*$ as kG-modules. For kG-modules M and N, we denote $\dim_k[\mathrm{Hom}_{kG}(M, N)]$ by $[M, N]^G$. For simple kG-modules S and T, $c(S, T)$ denotes the Cartan invariant with respect to S and T (usually, a notation $c_{S,T}$ is used instead). Other notation and terminology follow the books of Feit [1] and Landrock [3].

1. Calculation of determinants

In this section we give several lemmas on elementary computation of determinants for matrices over a field of characteristic 2. Throughout this section all entries of matrices are elements in a field of characteristic 2. This assumption is essential here.

Lemma 1.1. *For non-negative integers m and n, let A be a square matrix of size $m + 2n + 2$ of the form*

$$
A = \begin{array}{c} \\ 1 \\ \vdots \\ m \\ 0 \\ 0^* \\ \vdots \\ n \\ n^* \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} 1 & \dots & m & 1 & 1^* & \dots & n & n^* & u & v \end{array} \\
\left(\begin{array}{cccccccccc}
 & & & a_{11} & a_{11} & \dots & a_{1n} & a_{1n} & 0 & 0 \\
 & S & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & a_{m1} & a_{m1} & \dots & a_{mn} & a_{mn} & 0 & 0 \\
b_{01} & \dots & b_{0m} & c_{01} & d_{01} & & c_{0n} & d_{0n} & u_0 & v_0 \\
b_{01} & \dots & b_{0m} & d_{01} & c_{01} & \cdots & d_{0n} & c_{0n} & u_0 & u_0 \\
 & \vdots \quad \vdots & & & \vdots & \vdots & \vdots & & \vdots & \\
b_{n1} & \dots & b_{nm} & c_{n1} & d_{n1} & & c_{nn} & d_{nn} & u_n & v_n \\
b_{n1} & \dots & b_{nm} & d_{n1} & c_{n1} & \cdots & d_{nn} & c_{nn} & u_n & v_n
\end{array}\right)
\end{array}
$$

where S is an $(m \times m)$-matrix. Then, $\det A = 0$.

Proof. We prove the lemma by induction on n. If $n = 0$, then $\det A = 0$ since two rows indexed by 0 and 0^* are the same. Assume $n \geqslant 1$. Expand A with respect to the last column indexed by v. Then, by symmetry, it is enough to show that the determinant of the following square matrix B of size $m+2n+1$ is zero:

$$B = \begin{array}{c|ccccccccc}
 & 1 & \dots & m & 1 & 1^* & \dots & n & n^* & u \\ \hline
1 & & & & a_{11} & a_{11} & & a_{1n} & a_{1n} & 0 \\
\vdots & & S & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
m & & & & a_{m1} & a_{m1} & & a_{mn} & a_{mn} & 0 \\
0+0^* & 0 & \dots & 0 & e_{01} & e_{01} & \dots & e_{0n} & e_{0n} & 0 \\
1 & b_{11} & \dots & b_{1m} & c_{11} & d_{11} & & c_{1n} & d_{1n} & u_1 \\
1^* & b_{11} & \dots & b_{1m} & d_{11} & c_{11} & \cdots & d_{1n} & c_{1n} & u_1 \\
\vdots & & \vdots \quad \vdots & & & \vdots & \vdots & & \vdots & \vdots \\
n & b_{n1} & \dots & b_{nm} & c_{n1} & d_{n1} & & c_{nn} & d_{nn} & u_n \\
n^* & b_{n1} & \dots & b_{nm} & d_{n1} & c_{n1} & \cdots & d_{nn} & c_{nn} & u_n
\end{array}$$

where $e_{0i} = c_{0i} + d_{0i}$ for $i = 1, ..., n$. Now, expand B with respect to the final column indexed by u. So, again by symmetry, it suffices to prove $\det C = 0$ for the following square matrix C of size $m+2n$:

$$C = \begin{array}{c|cccccccc}
 & 1 & \dots & m & 1 & 1^* & \dots & n & n^* \\ \hline
1 & & & & a_{11} & a_{11} & & a_{1n} & a_{1n} \\
\vdots & & S & & \vdots & \vdots & \vdots & \vdots & \vdots \\
m & & & & a_{m1} & a_{m1} & & a_{mn} & a_{mn} \\
0+0^* & 0 & \dots & 0 & e_{01} & e_{01} & \dots & e_{0n} & e_{0n} \\
1+1^* & 0 & \dots & 0 & e_{11} & e_{11} & \dots & e_{1n} & e_{1n} \\
2 & b_{21} & \dots & b_{2m} & c_{21} & d_{21} & & c_{2n} & d_{2n} \\
2^* & b_{21} & \dots & b_{2m} & d_{21} & c_{21} & \cdots & d_{2n} & c_{2n} \\
\vdots & & \vdots \quad \vdots & & & \vdots & \vdots & & \vdots \\
n & b_{n1} & \dots & b_{nm} & c_{n1} & d_{n1} & & c_{nn} & d_{nn} \\
n^* & b_{n1} & \dots & b_{nm} & d_{n1} & c_{n1} & \cdots & d_{nn} & c_{nn}
\end{array}$$

where $e_{1i} = c_{1i} + d_{1i}$ for $i = 1, ..., n$. Then, by carrying two rows indexed by $0+0^*$ and $1+1^*$ to the end, and then by taking its transpose, we have $\det C = 0$ by induction. ☐

Lemma 1.2. *For non-negative integers m and n, let A be a square matrix of size $m+2n+1$ of the form*

$$A = \begin{array}{c|ccccccccc}
 & u & 1 & \dots & m & 1 & 1^* & \dots & n & n^* \\ \hline
v & 0 & 0 & \dots & 0 & v_1 & v_1 & \dots & v_n & v_n \\
1 & 0 & & & & a_{11} & a_{11} & & a_{1n} & a_{1n} \\
\vdots & \vdots & & S & & \vdots & \vdots & \vdots & \vdots & \vdots \\
m & 0 & & & & a_{m1} & a_{m1} & & a_{mn} & a_{mn} \\
1 & u_1 & b_{11} & \dots & b_{1m} & c_{11} & d_{11} & & c_{1n} & d_{1n} \\
1^* & u_1 & b_{11} & \dots & b_{1m} & d_{11} & c_{11} & \cdots & d_{1n} & c_{1n} \\
\vdots & \vdots & & \vdots \quad \vdots & & & \vdots & \vdots & & \vdots \\
n & u_n & b_{n1} & \dots & b_{nm} & c_{n1} & d_{n1} & & c_{nn} & d_{nn} \\
n^* & u_n & b_{n1} & \dots & b_{nm} & d_{n1} & c_{n1} & \cdots & d_{nn} & c_{nn}
\end{array}$$

where S is an $(m \times m)$-matrix. Then, $\det A = 0$.

Proof. First of all, expand A with respect to the first row indexed by v, and sum each pair of determinants of size $m+2n$ that have the same coefficient v_i for $i=1,...,n$. Namely, we can write $\det A = \sum_{i=1}^{n} v_i \cdot \det B_i$, where each B_i is a square matrix of size $m+2n$ and each B_i has the same form as in Lemma 1.1 by a suitable exchanging of columns. Therefore, $\det A = 0$ by Lemma 1.1. □

2. Lemmas

In this section we state several lemmas which will be used in the proofs of our results. Throughout this section we assume that k is an algebraically closed field of characteristic $p>0$, and we fix a finite group G such that p divides the order of G.

Lemma 2.1 (Webb [9, Theorem E]). *Let $P=P(k_G)$ and assume $j(P)\geqslant 3$. If p is odd, then $P\cdot J(kG)/\mathrm{Soc}(P)$ is an indecomposable kG-module.*

Lemma 2.2 ([6, Lemma 1.2]). *If M is an indecomposable kG-module with $j(M)=2$, then $M\cdot J(kG)=\mathrm{Soc}(M)$.*

Lemma 2.3 ([3, II Corollary 6.9]). *For kG-modules M and N,*

$$[M,N]^G = [N^*,M^*]^G.$$

Lemma 2.4 ([3, I Lemma 8.4 (i)]). *For a kG-module M,*

$$[M/(M\cdot J(kG))]^* \simeq \mathrm{Soc}(M^*) \qquad \text{as } kG\text{-modules.}$$

Lemma 2.5 (Landrock). *For simple kG-modules S and T,*

$$c(S,T)=c(T,S)=c(S^*,T^*)=c(T^*,S^*).$$

Proof. We get the assertion from [1, I Lemma 14.9 and Theorem 16.7] and Landrock's result [4, Theorem A] (cf. [3, I Theorem 9.9]). □

3. Proofs

In this section we give proofs of the theorem and the corollary in the introduction.

Proof of Theorem. Let B be a block ideal of kG containing S. Since S is self-dual, Y^* is a simple kG-module in B again if Y is a simple kG-module in B (see [3, I Proposition 10.8]). Thus, let $S_0=S,S_1,...,S_m,T_1,{T_1}^*,...,T_n,{T_n}^*$ all be non-isomorphic simple kG-modules in B; and $S_i\simeq {S_i}^*$ for all i and $T_j \not\simeq {T_j}^*$ for all j. We denote by C the Cartan matrix for B, and let $\overline{C}$ be its image induced by the canonical epimorphism $\mathbb{Z}\to\mathbb{Z}/2\mathbb{Z}$.

Now, suppose that $c(S_0,S_i)$ is even for all $i=0,...,m$. Then Lemma 2.5 implies that $\overline{C}$ has the same form as in Lemma 1.2, so that $\det\overline{C}=0$ from Lemma 1.2. This means $\det C$ is even, which contradicts Brauer's result [1, IV Theorem 3.9]). □

Proof of Corollary. First of all, we may assume that k is algebraically closed (see [5, Proposition 12.11]). Let $J=J(kG)$, $P=P(k_G)$ and $M=PJ/\mathrm{Soc}(P)$.

Assume p is odd. By Lemmas 2.1 and 2.2, M is an indecomposable kG-module with $MJ=\mathrm{Soc}(M)$. Then, the Theorem implies that there is a simple kG-module T such that T is self-dual and $c(k_G,T)$ is odd. On the other hand, since T and M are both self-dual, and since $MJ=\mathrm{Soc}(M)$, it follows from Lemmas 2.3 and 2.4 that

$$[M/MJ,T]^G=[T,(M/MJ)^*]^G=[T,\mathrm{Soc}(M)]^G=[T,MJ]^G,$$

which says that the multiplicities of T in M/MJ and MJ as direct summands are the same, so that $c(k_G, T)$ is even, a contradiction. □

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