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INEQUALITIES FOR THE NOVIKOV-SHUBIN INVARIANTS

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Abstract. In this paper, we prove that the Novikov-Shubin invariants satisfy a sequence of inequalities and deduce some useful consequences of this result.

Introduction

Let (M, g) be a compact connected Riemannian manifold and $\widetilde{M}$ be its universal cover, which is a Riemannian manifold with the induced metric. Although we only consider the universal cover in this paper, our results extend easily to any Galois covering of M. Let Δ_j denote the Laplacian acting on L^2 differential j-forms on $\widetilde{M}$.

Let $k_j(t, x, y)$ denote the integral kernel of the heat operator $e^{-t\Delta_j}$. By parabolic regularity theory, $k_j(t, x, y)$ is smooth. Since Δ_j commutes with the action of the fundamental group $\pi_1(M)$, we see that

$$k_j(t, \gamma.x, \gamma.y) = k_j(t, x, y)$$

for all $\gamma \in \pi_1(M)$ (here we identify the cotangent spaces at x and $\gamma.x$ via γ). Recall that the von Neumann trace of $e^{-t\Delta_j}$ is given by

$$\tau(e^{-t\Delta_j}) = \int_M tr(k_j(t, x, x))dx.$$

Atiyah [1] defined the L^2 Betti numbers of M as

$$\beta_j = \tau(P_j),$$

where P_j denotes the orthogonal projection onto the kernel of Δ_j. Then, since $k_j(t, x, y)$ converges as $t \to \infty$ uniformly over compact subsets to $k_{P_j}(x, y)$, where $k_{P_j}(x, y)$ denotes the integral kernel of the operator P_j (cf. [12], proof of Theorem 13.14), we see that

$$\beta_j = \lim_{t\to\infty} \tau(e^{-t\Delta_j}).$$

Define the von Neumann algebra theta functions

$$\theta_j(t) = \tau(e^{-t\Delta_j}) - \beta_j.$$

We next recall the definition of the Novikov-Shubin invariants [4]:

$$\alpha_j = \alpha_j(M) = \sup\{\beta \in \mathbb{R} : \theta_j(t) \quad \text{is} \quad O(t^{-\beta}) \;\text{ as }\; t \to \infty\} \in [0, \infty],$$

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$$\overline{\alpha}_j = \overline{\alpha}_j(M) = \inf\{\beta \in I\!R : t^{-\beta} \quad \text{is} \quad O(\theta_j(t))) \ \text{ as } \ t \to \infty\} \in [0, \infty].$$

The Novikov-Shubin invariants are homotopy invariants of M [5], [3] and they satisfy Poincaré duality, that is, $\alpha_j(M) = \alpha_{n-j}(M)$ and $\overline{\alpha}_j(M) = \overline{\alpha}_{n-j}(M)$, where n denotes the dimension of the manifold M. This is easily verified using the Hodge star operator. In this paper, we establish some basic inequalities between the Novikov-Shubin invariants, as in the theorem below.

Theorem 0.1. *The Novikov-Shubin invariants satisfy the sequences of inequalities*

$$\text{minimum}\{\alpha_k, \alpha_{k-2}, ...\} \le \text{minimum}\{\alpha_{k-1}, \alpha_{k-3}, ...\}$$

and

$$\text{minimum}\{\overline{\alpha}_k, \overline{\alpha}_{k-2}, ...\} \le \text{minimum}\{\overline{\alpha}_{k-1}, \overline{\alpha}_{k-3}, ...\}$$

for $k = 1, 2, ..., n-1$, *with equality holding when* $k = n$.

The proof of this theorem will be given in the next section.

We now mention some consequences of this theorem. From the theorem we see that we always have the inequality

$$\alpha_0 \ge \alpha_1 \ge \text{minimum}\{\alpha_0, \alpha_2\}.$$

When $n = 1$, $\alpha_0 = \alpha_1$; when $n = 2$, $\alpha_0 = \alpha_1 = \alpha_2$; when $n = 3$, $\alpha_1 = \alpha_2$; and when $n = 4$, $\alpha_1 = \text{minimum}\{\alpha_0, \alpha_2\}$.

Recall the definition of positive decay for manifolds [10].

Definition. A closed manifold M is said to have positive decay if all its Novikov-Shubin invariants $\alpha_j(M) > 0$ are positive, for j satisfying $0 \le j \le \dim(M)$.

By the results of Gromov and Shubin [5], any closed manifold which is homotopy equivalent to one with positive decay also has positive decay. An important unsolved conjecture is that every closed manifold has positive decay. Our interest in this conjecture stems from our closely related work on L^2 torsion invariants [9, 2]; see also [6, 8] and the more general shifted L^2 torsion invariants [11]. These L^2 invariants are defined whenever the Novikov-Shubin invariants are positive.

The following are examples of closed manifolds with positive decay:

(1) Any closed, flat manifold.
(2) Any closed, hyperbolic manifold.
(3) Any closed, reductive locally symmetric space.
(4) Any closed manifold with virtually Abelian fundamental group.
(5) Any closed manifold whose fundamental group is a free product of virtually Abelian fundamental groups.
(6) Any non-exceptional closed, 3-dimensional manifold.
(7) Any Kähler hyperbolic or quaternionic Kähler hyperbolic manifold.

Details of these and other examples can be found in [9, 10] and [6, 7].

An immediate consequence of Theorem 0.1 and Poincaré duality is the following

Corollary 0.2. *(1) Let M be a closed manifold of dimension $2n$ or $2n+1$. If the even-degree Novikov-Shubin invariants of M, $\alpha_{2j}(M) > 0$, are positive for $0 \le j \le [\frac{n}{2} + 1]$, then M has positive decay.*

(2) Let M be a closed manifold of dimension $2n$ or $2n+1$. If the odd-degree Novikov-Shubin invariants of M, $\alpha_{2j+1}(M) > 0$, are positive for $0 \le j \le [\frac{n}{2} + 1]$, then M has positive decay.

1. The proofs

Hypothesis ($*$)**.** Assume that $\varphi, \varphi_\varepsilon$ are smooth, rapidly decreasing, positive functions on $[0,\infty)$ satisfying $\varphi(0) = \varphi_\varepsilon(0) = 1$, $0 < \varphi_\varepsilon \leq \varphi$ and $0 < \varphi'_\varepsilon(0) < \varphi'(0)$.

Since $\varphi - \varphi_\varepsilon$ is a smooth and rapidly decreasing positive function on $I\!R^+$ which vanishes at $\lambda = 0$, we can write

$$(\varphi - \varphi_\varepsilon)(\lambda) = \lambda \psi^2(\lambda),$$

where ψ is a positive, rapidly decreasing function on $[0,\infty)$.

We define the numbers

$$\begin{aligned} \mu_j &= \tau(\varphi(\Delta_j)), \\ \beta_j^\varepsilon &= \tau(\varphi_\varepsilon(\Delta_j)). \end{aligned}$$

Proposition 1.1. *The inequalities*

$$\sum_{j=0}^{k} (-1)^{k-j} \mu_j \geq \sum_{j=0}^{k} (-1)^{k-j} \beta_j^\varepsilon$$

hold for $k = 0, 1, \ldots, n-1$, with equality when $k = n$.

Proof. We can write

$$(\varphi - \varphi_\varepsilon)(\Delta_j) = \Delta_j \psi^2(\Delta_j).$$

Now since $\Delta_j = d\delta + \delta d$,

$$\begin{aligned} \tau(d\delta\psi^2(\Delta_j)) &= \tau(\psi(\Delta_j) d\delta \psi(\Delta_j)) \\ &= \tau(\delta \psi^2(\Delta_j) d) \\ &= \tau(\delta d \psi^2(\Delta_{j-1})). \end{aligned}$$

Therefore

$$\sum_{j=0}^{k} (-1)^{k-j} (\mu_j - \beta_j^\varepsilon) = \tau(\delta d \psi^2(\Delta_k)).$$

If $k = n$, the right-hand side is zero. In general,

$$\delta d \psi^2(\Delta_k) = A^* A \geq 0,$$

where $A = d\psi(\Delta_k)$. Therefore

$$\tau(\delta d \psi^2(\Delta_k)) = \tau(A^* A) \geq 0,$$

proving the proposition.

Now let $\varphi(\lambda) = e^{-t\lambda}$ and $\varphi_\varepsilon(\lambda) = e^{-\frac{t}{\varepsilon}\lambda}$ for $0 < \varepsilon \leq 1$. Then φ and φ_ε satisfy the hypotheses ($*$), that is, $\varphi(0) = \varphi_\varepsilon(0) = 1$, $0 < \varphi_\varepsilon \leq \varphi$ and $0 < \varphi'_\varepsilon(0) < \varphi'(0)$. Also $\varphi, \varphi_\varepsilon$ are smooth, rapidly decreasing, positive functions on $[0,\infty)$.

Proposition 1.2. *The inequalities*

$$\sum_{j=0}^{k} (-1)^{k-j} \mu_j \geq \sum_{j=0}^{k} (-1)^{k-j} \beta_j$$

hold for $k = 0, 1, \ldots, n-1$, with equality when $k = n$.

Proof. By Proposition 1.1, it suffices to prove that

$$\lim_{\varepsilon\to 0}\beta_j^{\varepsilon} = \beta_j \ .$$

Now

$$\begin{aligned}\lim_{\varepsilon\to 0}\beta_j^{\varepsilon} &= \lim_{\varepsilon\to 0}\tau(e^{-\frac{t}{\varepsilon}\Delta_j})\\ &= \tau(P_j) = \beta_j \ ,\end{aligned}$$

since the heat kernel $k_j(\frac{t}{\varepsilon}, x, y)$ converges as $\varepsilon \to 0$ uniformly over compact sets to the integral kernel $k_{P_j}(x, y)$ of the orthogonal projection P_j onto the L^2 harmonic j-forms (cf. [12], proof of Theorem 13.14).

Proof of Theorem 0.1*.* By Proposition 1.2, it follows that

$$\theta_k(t) + \theta_{k-2}(t) + \theta_{k-4}(t) + ... \geq \theta_{k-1}(t) + \theta_{k-3}(t) + ... \ .$$

This immediately yields the inequalities stated in the theorem.

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