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SPECTRALLY DETERMINED GROWTH IS GENERIC

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ABSTRACT. Let A be the infinitesimal generator of a C_0-semigroup of operators in a Hilbert space. We consider the class of operators $A+B$, where B is bounded. It is proved that the spectrum of $A+B$ determines the growth of the associated semigroup for "most" operators B (in the sense of Baire category).

1. INTRODUCTION

As is well known, the stability of the system $\dot{y} = Ay$ of linear ODEs with constant coefficients is determined by the eigenvalues of the matrix A. In infinite dimensions, the issue is more complex. The appropriate abstract setting is that of C_0-semigroups of linear operators in a Banach space X. Let A be an infinitesimal generator, and let $\exp(At)$ be the associated semigroup. As usual, we define the type of the semigroup by

$$\omega(A) = \lim_{t\to\infty} \log \|e^{At}\|/t, \tag{1}$$

and we define the spectral bound as

$$r(A) = \sup\{\mathrm{Re}\ \lambda \,|\, \lambda \in \sigma(A)\}, \tag{2}$$

where $\sigma(A)$ denotes the spectrum of A.

It is well known that $\omega(A) \geq r(A)$. However, equality is not true in general. A first counterexample was given in [1]. Zabczyk [4] has given a much more natural example, and his ideas were recently used to show that counterexamples can be as simple and natural as a lower order perturbation of the wave equation [3]. This, in particular, destroys all hopes that "applied" problems are somehow well-behaved enough to avoid "pathological" cases where $\omega(A)$ is strictly greater than $r(A)$. This is disturbing, since computation of the spectrum is the most widely used practical method for assessing the stability of physical systems.

In this note, we take a different view. Rather than considering one operator A, we shall look at the class of operators $A+B$, where B ranges over the set $B(X)$ of all bounded linear operators in X. Let us define

$$S(A) = \{B \in B(X) \,|\, \omega(A+B) > r(A+B)\}. \tag{3}$$

We shall prove the following result.

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Theorem. *If X is a Hilbert space, then the set $S(A)$ is of first category in $B(X)$ (with the topology of the operator norm).*

In a sense this theorem asserts that the spectrum determines the growth of the semigroup "most of the time".

2. Proof of the theorem

To prove the theorem, we shall first establish a few lemmas.

Lemma 1. *Let C be a closed operator in X with bounded inverse. Assume that $\|C^{-1}\| > 1/\epsilon$. Then there exists $D \in B(X)$ with $\|D\| \le 2\epsilon$ such that 0 is a simple isolated eigenvalue of $C + D$.*

Proof. It follows from the assumptions that there exists $x_0 \in D(C)$ such that $\|x_0\| = 1$ and $\|Cx_0\| \le \epsilon$. By the Hahn-Banach theorem we can pick $y_0 \in X^*$ such that $\|y_0\| = 1$ and $(y_0, x_0) = 1$. By choosing y_1 to be a small perturbation of y_0, we can arrange it so that $\|y_1\| \le 3/2$, $(y_1, x_0) = 1$ and $(y_1, C^{-1}x_0) \ne 0$. Let us now define

$$D_1 x = -(y_1, x)Cx_0. \tag{4}$$

Then $\|D_1\| \le \epsilon\|y_1\| \le 3\epsilon/2$. Since D_1 has finite rank, $C + D_1$ has Fredholm index zero. Moreover,

$$(C + D_1)x_0 = 0,\ (y_1, C^{-1}(C + D_1)x) = 0. \tag{5}$$

Let us define the subspace

$$X_1 = \{x \in X \mid (y_1, C^{-1}x) = 0\}. \tag{6}$$

Since X_1 is the nullspace of a linear functional, it has codimension one. We can decompose $X = X_1 + [x_0]$, where $[x_0]$ denotes the linear span of x_0. One easily checks that this decomposition is also continuous in $D(C)$ (equipped with the usual graph norm). Now consider the restriction of $C + D_1$ to $D(C) \cap X_1$, viewed as an operator in X_1. Since $(C + D_1)x_0 = 0$, the range of the restriction is the same as the range of $C + D_1$, but changing the space from X to X_1 has reduced the codimension by 1. Since x_0 is in the nullspace of $C + D_1$, the restriction also reduces the dimension of the nullspace by one. Hence the restricted operator still has Fredholm index zero, and hence there exists an arbitrarily small perturbation $D_2 \in B(X_1)$ such that $C + D_1 + D_2$ is invertible on X_1. We extend D_2 to all of X by setting $D_2 x_0 = 0$. The operator $D = D_1 + D_2$ is the one we seek. □

We now define $Q(A, \epsilon)$ to be the set of all $B \in B(X)$ such that $A + B$ has a simple isolated eigenvalue with real part greater than $\omega(A + B) - \epsilon$.

Lemma 2. *$Q(A, \epsilon)$ is open.*

Proof. Let $B \in Q(A, \epsilon)$. Then there exists $\delta > 0$ such that $A + B$ has a simple isolated eigenvalue with real part greater than $\omega(A + B) - \epsilon + \delta$. If $\|B_1\|$ is small enough, then $\omega(A + B + B_1) \le \omega(A + B) + \delta/2$, and $A + B + B_1$ has a simple isolated eigenvalue within distance $\delta/2$ of the isolated eigenvalue of $A + B$. This implies that $A + B + B_1 \in Q(A, \epsilon)$, and hence $Q(A, \epsilon)$ is open. □

Lemma 3. *Assume that X is a Hilbert space. Then $Q(A, \epsilon)$ is dense in $B(X)$ with the norm topology.*

Proof. Let $B \in B(X)$ and $\delta > 0$ be given. By a theorem of Prüß [2], the resolvent of $A + B$ is unbounded in the halfplane $\operatorname{Re}\lambda > \omega(A+B) - \delta$. Moreover, there exists a constant M such that $\|\exp((A+B)t)\| \leq M \exp((\omega(A+B)+\delta)t)$. Now prescribe an arbitrary $\epsilon > 0$. Then by Lemma 1, there exists a perturbation B_1, with norm less than ϵ, such that $A + B + B_1$ has an isolated simple eigenvalue in the halfplane $\operatorname{Re}\lambda > \omega(A+B) - \delta$. Moreover, the type of $A + B + B_1$ is at most $\omega(A+B) + \delta + M\epsilon$. Thus $B + B_1 \in Q(A, 2\delta + M\epsilon)$. Since ϵ was arbitrary, we can choose it such that $M\epsilon < \delta$, so that $B + B_1 \in Q(A, 3\delta)$. □

The theorem now follows by observing that $S(A)$ lies in the complement of

$$\bigcap_{n=1}^{\infty} Q(A, \frac{1}{n}). \tag{7}$$

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