PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 8, August 1996

INTEGRAL INCLUSIONS OF UPPER SEMI-CONTINUOUS OR LOWER SEMI-CONTINUOUS TYPE

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Topological results for set valued maps are used to establish existence results for integral inclusions of Volterra or Hammerstein type.

1. INTRODUCTION

This paper presents existence results for Volterra integral inclusions of the form

$$y(t) \in \int_0^t k(t,s)F(s,y(s))ds + g(t), \qquad t \in [0,T], \tag{1.1}$$

and Hammerstein integral inclusions of the form

$$y(t) \in \int_0^T k(t,s)F(s,y(s))ds + g(t), \qquad t \in [0,T], \tag{1.2}$$

where $F : [0,T] \times \mathbf{R} \to \mathbf{R}$ is a multivalued function with nonempty compact values. Here $T > 0$ is a constant. Throughout this paper the map $x \mapsto F(t,x)$ is either upper semi-continuous or lower semi-continuous for a.e. $t \in [0,T]$.

The study of integral equations has received much attention over the last fifty years or so. However very few results are available for integral inclusions; see [2]; [3, chapter 4]; [7, chapter 12] and their references. Our paper has two main objectives. We first show that the techniques applied in establishing existence results for integral equations [3, 4, 7, 9, 10] transfer naturally to integral inclusions. Our ideas were motivated by results of Pruszko [11, 12] (for second order differential inclusions) and more recently by results of Frigon and Granas [6]. Our second objective is to use these ideas to establish existence results for superlinear integral inclusions; these results are new even for integral equations.

The paper will be divided into three main sections. Section 2 gathers together some known results on multivalued maps. We will then use these results in section 3 to establish existence principles for the integral inclusions (1.1) and (1.2). Existence theory for sublinear and superlinear integral inclusions will be established in section 4.

Received by the editors February 6, 1995.
1991 *Mathematics Subject Classification.* Primary 45D05, 45G10.

2. Preliminaries

Let E_1, E_2 be two Banach spaces, X a nonempty closed subset of E_1 and S a measurable space (respectively $S = I \times \mathbf{R}^n$, where I is a real interval, and $A \subseteq S$ is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $N \times D$ where N is Lebesgue measurable in I and D is Borel measurable in $\mathbf{R}^n$). Let $H : X \to E_2$ and $G : S \to E_2$ be two multifunctions with nonempty closed values. The function G is *measurable* (respectively $\mathcal{L} \otimes \mathcal{B}$ measurable) if the set $\{t \in S : G(t) \cap B \neq \varnothing\}$ is measurable for any closed set B in S. A subset A of $L^1([0,T], \mathbf{R}^n)$ is *decomposable* if for all $u, v \in A$ and $N \subseteq [0,T]$ measurable the function $u\chi_N + v\chi_{[0,T]/N} \in A$.

The function H is *lower semi-continuous* (l.s.c.) (respectively *upper semi-continuous* (u.s.c.)) if the set $\{x \in X : H(x) \cap B \neq \varnothing\}$ is open (respectively closed) for any open (respectively closed) set B in E_2. If H is l.s.c. and u.s.c., then H is continuous. H is *compact* if $\overline{H(X)} = \overline{\bigcup_{x \in X} H(x)}$ is compact in E_2 and it is *completely continuous* if $\overline{H(\Omega)}$ is compact for all bounded sets $\Omega \subseteq X$.

H is called *weakly upper semi-continuous* (w-u.s.c.) if for all $\{x_n\} \subseteq X$ and $\{y_n\} \subseteq E_2$ the condition

$$x_n \to x,\ y_n \rightharpoonup y,\ y_n \in H(x_n) \quad \text{for all } n$$

implies $y \in H(x)$.

Remark. Here $\rightharpoonup$ denotes weak convergence.

H is called *weakly completely continuous* if H is w-u.s.c. and for every bounded subset Ω of X, $H(\Omega) = \bigcup_{a \in \Omega} H(a)$ is a relatively weakly compact subset of E_2.

We now state without proof (see [11, 12]) two results.

Theorem 2.1. *Let $H : X \to E_2$ be a completely continuous multivalued map. Then H is u.s.c. if and only if the graph $\mathcal{G}(H) = \{(x,y) \in X \times E_2 : y \in H(x)\}$ is a closed subset of $X \times E_2$.*

Theorem 2.2. *Let E be any Banach space. Suppose $H : X \to E_2$ is a weakly completely continuous multivalued mapping and $T : E_2 \to E$ is given by $T(x) = T_1(x) + p$, $x \in E_2$, where p is a (fixed) element of E and $T_1 : E_2 \to E$ is a continuous linear mapping. Then the graph $\mathcal{G}(T \circ H)$ of the mapping $T \circ H : X \to E$ is a closed subset of $X \times E$.*

Remark. The proof of Theorem 2.2 is immediate from the above definitions, see also [11, Proposition 1.5].

Let $F : [0,T] \times \mathbf{R}^n \to \mathbf{R}^n$ be a multivalued function with nonempty compact values. Throughout this paper our map F will satisfy **some** of the following properties (to be specified later):

$$\begin{cases} \text{(i) } (t,x) \mapsto F(t,x) \text{ is } \mathcal{L} \otimes \mathcal{B} \text{ measurable,} \\ \text{(ii) } x \mapsto F(t,x) \text{ is l.s.c. for a.e. } t \in [0,T], \end{cases} \tag{2.1}$$

$$\begin{cases} \text{(i) } t \mapsto F(t,x) \text{ is measurable for every } x \in \mathbf{R}^n, \\ \text{(ii) } x \mapsto F(t,x) \text{ is continuous for a.e. } t \in [0,T], \end{cases} \tag{2.2}$$

$$\begin{cases} \text{(i) } t \mapsto F(t,x) \text{ is measurable for every } x \in \mathbf{R}^n, \\ \text{(ii) } x \mapsto F(t,x) \text{ is u.s.c. for a.e. } t \in [0,T], \end{cases} \tag{2.3}$$

$$\text{(2.4)} \qquad \begin{cases} \text{for each } r > 0 \text{ there exists a function } h_r \in L^1[0,T] \text{ such that} \\ |F(t,x)| \le h_r(t) \text{ for a.e. } t \in (0,T) \text{ and every } x \in \mathbf{R}^n \text{ with } |x| \le r. \end{cases}$$

Assign to F a multivalued operator

$$\mathcal{F} : C[0,T] \to L^1[0,T]$$

by letting

$$\text{(2.5)} \qquad \mathcal{F}(y) = \{w \in L^1[0,T] : w(t) \in F(t, y(t)) \text{ for a.e. } t \in (0,T)\}.$$

Theorem 2.3 ([11]). *Let $F : [0,T] \times \mathbf{R}^n \to \mathbf{R}^n$ satisfy (2.3) and (2.4). Then the multivalued mapping $\mathcal{F} : C[0,T] \to L^1[0,T]$ defined by (2.5) is weakly completely continuous and integrably bounded on bounded sets.*

Remark. $\mathcal{F}$ is integrably bounded on bounded sets if for every bounded set A there exists $h_A \in L^1[0,T]$ such that for every $x \in A$ and $w \in \mathcal{F}(x)$ we have $|w(t)| \le h_A(t)$ a.e. on $(0,T)$.

Next we state a recent selection theorem [1] due to Bressan and Colombo. Let Y be a metric space and $G : Y \to L^1[0,T]$ a multivalued operator. We say G has property (BC) if

(i) G is l.s.c.

(ii) G has nonempty closed and decomposable values.

Theorem 2.4 ([1]). *Let Y be a separable metric space and let $G : Y \to L^1[0,T]$ be a multivalued operator which has property (BC). Then G has a continuous selection, i.e. there exists a continuous function (single valued) $g : Y \to L^1[0,T]$ such that $g(y) \in G(y)$ for every $y \in Y$.*

Finally we state the two fixed point results which will be used in section 3.

Theorem 2.5 (Nonlinear alternative for single valued maps [5, 8]). *Assume U is a relatively open subset of a convex set K in a Banach space E. Let $N : \overline{U} \to K$ be a compact map with $p \in U$. Then either*

(i) *N has a fixed point in $\overline{U}$; or*

(ii) *there is a $u \in \partial U$ and a $\lambda \in (0,1)$ such that $u = \lambda N u + (1-\lambda)p$.*

Theorem 2.6 (Nonlinear alternative for multivalued maps [5]). *Let K be a convex subset of a Banach space, $U \subseteq K$ be relatively open, and $p \in U$. Suppose $N : \overline{U} \to K$ is an u.s.c. compact multivalued map with nonempty, compact, convex values. Then either*

(i) *there is a $u \in \overline{U}$ such that $u \in Nu$; or*

(ii) *there is a $u \in \partial U$ and a $\lambda \in (0,1)$ such that $u \in \lambda N u + (1-\lambda)p$.*

3. Existence principles

We begin by presenting some existence principles for the Volterra integral inclusion

$$\text{(3.1)} \qquad y(t) \in \int_0^t k(t,s) F(s, y(s))ds + g(t), \qquad t \in [0,T].$$

Now $F : [0,T] \times \mathbf{R} \to \mathbf{R}$ is a multivalued function with nonempty compact values. The kernel k satisfies the following conditions:

$$\text{(3.2)} \qquad \begin{cases} \text{for each } t \in [0,T],\ k(t,s) \text{ is measurable on } [0,t] \text{ and} \\ k(t) = \text{ess sup}\, |k(t,s)|, 0 \le s \le t, \text{ is bounded on } [0,T], \end{cases}$$

$$\begin{cases} \text{the map } t \mapsto k_t \text{ is continuous from } [0,T] \text{ to } L^\infty[0,T]; \\ \text{here } k_t(s) = k(t,s). \end{cases} \tag{3.3}$$

Now assume $g : [0,T] \to \mathbf{R}$ is single valued with

$$g \in C[0,T]. \tag{3.4}$$

Define the operator

$$S : L^1[0,T] \to C[0,T] \tag{3.5}$$

by

$$Sh(t) = \int_0^t k(t,s)h(s)ds + g(t). \tag{3.6}$$

To see that the above makes sense let $h \in L^1[0,T]$. Notice first that

$$\left| \int_0^t k(t,s)h(s)ds \right| \le \left(\sup_{[0,T]} k(t) \right) \int_0^T |h(s)|\, ds.$$

Also for $t, \tau \in [0,T]$ with $t < \tau$ we have

$$\begin{aligned} |Sh(t) - Sh(\tau)| &= \left| \int_0^t k(t,s)h(s)ds - \int_0^\tau k(\tau,s)h(s)\, ds \right| \\ &= \left| \int_0^\tau [k(t,s) - k(\tau,s)]h(s)ds + \int_\tau^t k(t,s)h(s)ds \right| \\ &\le |k(t,\cdot) - k(\tau,\cdot)|_{L^\infty} \int_0^T |h(s)|ds + \left(\sup_{[0,T]} k(t) \right) \int_t^\tau |h(s)|\, ds. \end{aligned}$$

Lemma 3.1. *Suppose* (3.2), (3.3) *and* (3.4) *are satisfied. Then the operator* $S : L^1[0,T] \to C[0,T]$, *given by* (3.6), *is continuous.*

Proof. Let $h_1, h_2 \in L^1[0,T]$. Then

$$\begin{aligned} |Sh_1(t) - Sh_2(t)| &= \left| \int_0^t k(t,s)[h_1(s) - h_2(s)]ds \right| \\ &\le \left(\sup_{[0,T]} k(t) \right) \int_0^T |h_1(s) - h_2(s)|ds. \quad \square \end{aligned}$$

Define the multivalued operator

$$\mathcal{F} : C[0,T] \to L^1[0,T]$$

by

$$\mathcal{F}(y) = \{ w \in L^1[0,T] : w(t) \in F(t, y(t)) \text{ for a.e. } t \in (0,T) \}. \tag{3.7}$$

Definition. Let $F : [0,T] \times \mathbf{R} \to \mathbf{R}$ be a multivalued function with nonempty compact values. We say

(i) F is of *lower semi-continuous type* (l.s.c. type) if $\mathcal{F}$, given by (3.7), is l.s.c. and has nonempty, closed and decomposable values.

(ii) $S \circ F$ is of *upper semi-continuous type* (u.s.c. type) if $S \circ \mathcal{F}$, given in (3.6) and (3.7), is u.s.c., completely continuous and has nonempty, compact, convex values.

Theorem 3.2. *Let $F : [0,T] \times \mathbf{R} \to \mathbf{R}$ be a multivalued function with nonempty, compact, convex values. Assume* (2.3), (2.4), (3.2), (3.3) *and* (3.4) *are satisfied. Then $S \circ F$ is of u.s.c. type.*

Proof. Theorem 2.3 implies that $\mathcal{F} : C[0,T] \to L^1[0,T]$ is weakly completely continuous and integrably bounded on bounded sets. Notice for any $v \in L^1[0,T]$ that $Sv(t) = Sv_1(t)+g(t)$ where $S_1 : L^1[0,T] \to C[0,T]$ is the continuous linear operator given by

$$S_1 v(t) = \int_0^t k(t,s)v(s)ds.$$

Now Theorem 2.2 implies that $\mathcal{G}(S \circ \mathcal{F})$ is a closed subset of $C[0,T] \times L^1[0,T]$. In addition notice if Ω is any bounded subset of $C[0,T]$, then $S \circ \mathcal{F}(\Omega)$ is relatively compact. Now Theorem 2.1 implies that $S \circ \mathcal{F}$ is u.s.c. □

Theorem 3.3 ([6])**.** *Let $F : [0,T] \times \mathbf{R} \to \mathbf{R}$ be a multivalued function with nonempty, compact values. Assume* (2.4) *holds with F satisfying either* (2.1) *or* (2.2)*. Then F is of l.s.c. type.*

We now prove two existence principles for the integral inclusion (3.1).

Theorem 3.4. *Let $F : [0,T] \times \mathbf{R} \to \mathbf{R}$ be a multivalued function with nonempty, compact, convex values. Suppose* (2.3), (2.4), (3.2), (3.3) *and* (3.4) *are satisfied. In addition assume there is a constant M_0, independent of λ, with*

$$|y|_0 = \sup_{[0,T]} |y(t)| \neq M_0$$

for any solution y to

$$(3.8)_\lambda \qquad y(t) \in \lambda \left(\int_0^t k(t,s)F(s,y(s))ds + g(t) \right), \qquad t \in [0,T],$$

for each $\lambda \in (0,1)$. Then (3.1) *has a solution in $C[0,T]$.*

Proof. Solving $(3.8)_\lambda$ is equivalent to the fixed point problem $y \in S \circ \mathcal{F}(y) \equiv \lambda N y$ where $N = S \circ \mathcal{F}$ (here S is as in (3.6) and $\mathcal{F}$ is as in (3.7)). Notice from Theorem 3.2 that N is of u.s.c. type. Let

$$U = \{u \in C[0,T] : |u|_0 < M_0\} \quad \text{and} \quad E = K = C[0,T].$$

Apply Theorem 2.6 (nonlinear alternative for multivalued maps) and we may deduce the result immediately. □

Theorem 3.5. *Let $F : [0,T] \times \mathbf{R} \to \mathbf{R}$ be a multivalued function with nonempty, compact values. Suppose* (2.4), (3.2), (3.3) *and* (3.4) *are satisfied. Also assume F satisfies either* (2.1) *or* (2.2)*. In addition suppose there is a constant M_0, independent of λ, with $|y|_0 \neq M_0$ for any solution y to $(3.8)_\lambda$ for each $\lambda \in (0,1)$. Then* (3.1) *has a solution in $C[0,T]$.*

Proof. Solving $(3.8)_\lambda$ is equivalent to the fixed point problem $y \in \lambda N_1 y$ where $N_1 = S \circ \mathcal{F}$. Now Theorem 3.3 together with the Bressan Colombo selection theorem (Theorem 2.4) implies that F has a continuous selection $f : C[0,T] \to L^1[0,T]$. Consider the family of problems

$$(3.9)_\lambda \qquad y(t) = \lambda \left(\int_0^t k(t,s)f(s,y(s))ds + g(t) \right), \qquad t \in [0,T],$$

for $\lambda \in (0,1)$. By assumption, any solution y to $(3.9)_\lambda$ satisfies $|y|_0 \neq M_0$. Let

$$U = \{u \in C[0,T] : |u|_0 < M_0\} \quad \text{and} \quad E = K = C[0,T]$$

and

$$N : C[0,T] \to C[0,T] \text{ is defined by } Ny(t) = \int_0^t k(t,s)f(s,y(s))ds + g(t).$$

Remark. Notice N is continuous and completely continuous (follows immediately from the Arzela-Ascoli theorem and the analogue of the two displayed equations after (3.6)).

Apply Theorem 2.5 (nonlinear alternative for single valued maps) to deduce that $(3.9)_1$ has a solution. Consequently (3.1) has a solution. □

Next we present two existence principles for the Hammerstein integral inclusion

$$y(t) \in \int_0^T k(t,s)F(s,y(s))ds + g(t), \qquad t \in [0,T]. \tag{3.10}$$

Here $F : [0,T] \times \mathbf{R} \to \mathbf{R}$ is a multivalued function with nonempty compact values and $g : [0,T] \to \mathbf{R}$ is a single valued function. The kernel k satisfies

$$\begin{cases} \text{for each } t \in [0,T],\ k(t,s) \text{ is measurable on } [0,T] \text{ and} \\ k(t) = \text{ess sup}\,|k(t,s)|,\ 0 \leq s \leq T, \text{ is bounded on } [0,T]. \end{cases} \tag{3.11}$$

Essentially the same reasoning as in Theorems 3.4 and 3.5 establishes the following two existence principles for the integral inclusion (3.10).

Theorem 3.6. *Let $F : [0,T] \times \mathbf{R} \to \mathbf{R}$ be a multivalued function with nonempty, compact, convex values. Suppose* (2.3), (2.4), (3.3), (3.4) *and* (3.11) *are satisfied. In addition assume there is a constant M_0, independent of λ, with $|y|_0 \neq M_0$ for any solution y to*

$$y(t) \in \lambda\left(\int_0^T k(t,s)F(s,y(s))ds + g(t)\right), \qquad t \in [0,T], \tag*{$(3.12)_\lambda$}$$

for each $\lambda \in (0,1)$. Then (3.10) *has a solution in $C[0,T]$.*

Theorem 3.7. *Let $F : [0,T] \times \mathbf{R} \to \mathbf{R}$ be a multivalued function with nonempty, compact values. Suppose* (2.4), (3.3), (3.4) *and* (3.11) *are satisfied. Also assume F satisfies either* (2.1) *or* (2.2)*. In addition suppose there is a constant M_0, independent of λ, with $|y|_0 \neq M_0$ for any solution y to $(3.12)_\lambda$ for each $\lambda \in (0,1)$. Then* (3.10) *has a solution in $C[0,T]$.*

4. Existence theory

The usual existence results for integral equations of Volterra or Hammerstein type [3, 4, 9, 10] are easily extended (by essentially the same reasoning) to integral inclusions; consequently we will omit these results. Instead we now establish a result (which is new even in the single valued case) for the Hammerstein integral inclusion

$$y(t) \in \mu \int_0^T k(t,s)F(s,y(s))ds + g(t), \qquad t \in [0,T], \tag{4.1}$$

where $\mu \geq 0$ is a parameter.

Theorem 4.1. *Let $F : [0,T] \times \mathbf{R} \to \mathbf{R}$ be a multivalued function with nonempty, compact, convex values. Suppose* (2.3), (2.4), (3.3), (3.4) *and* (3.11) *are satisfied. In addition assume*

$$\text{(4.2)} \qquad \begin{cases} |F(t,y)| \le q(t) f(|y|) \text{ for almost all } t \in [0,T] \text{ and all } y \in \mathbf{R}^n; \text{ here} \\ f : [0,\infty) \to [0,\infty) \text{ is a Borel measurable function and } q \in L^1[0,T]. \end{cases}$$

Case (a): *Suppose*

$$\text{(4.3)} \qquad q \text{ is bounded on } [0,T]$$

and

$$\text{(4.4)} \qquad \begin{cases} \text{there exists a continuous function } f_1 : [0,\infty) \to [0,\infty) \text{ such that} \\ f_1(u) > 0 \text{ for } u > 0 \text{ and } \int_0^T f(|u(s)|)ds \le f_1(|u|_0) \text{ for any} \\ u \in C[0,T]; \text{ here } |u|_0 = \sup_{[0,T]} |u(t)| \end{cases}$$

hold. Let μ_0 satisfy

$$\text{(4.5)} \qquad \sup_{c\in[0,\infty)} \left(\frac{c}{\sup_{[0,T]} |g(t)| + \mu_0 f_1(c)(\sup_{[0,T]} k(t))(\sup_{[0,T]} q(t))} \right) > 1$$

where $k(t) = \operatorname{ess\,sup} |k(t,s)|$, $0 \le s \le T$. *If* $0 \le \mu \le \mu_0$, *then* (4.1) *has a solution.*

Case (b): *Suppose*

(4.6)

$$f \text{ in (4.2) is continuous and nondecreasing on } [0,\infty) \text{ with } f(u) > 0 \text{ for } u > 0$$

is satisfied. Let μ_0 satisfy

$$\text{(4.7)} \qquad \sup_{c\in[0,\infty)} \left(\frac{c}{\sup_{[0,T]} |g(t)| + \mu_0 f(c)(\sup_{[0,T]} k(t)) \int_0^T q(s)ds} \right) > 1.$$

If $0 \le \mu \le \mu_0$, *then* (4.1) *has a solution.*

Remark. The supremum in (4.5), (4.7) is allowed to be infinite.

Proof. Case (a): Fix $\mu \le \mu_0$. Let $M_0 > 0$ satisfy

$$\text{(4.8)} \qquad \frac{M_0}{\sup_{[0,T]} |g(t)| + \mu f_1(M_0)(\sup_{[0,T]} k(t))(\sup_{[0,T]} q(t))} > 1.$$

Let $y \in C[0,T]$ be a solution of

$$\text{(4.9)}_\lambda \qquad y(t) \in \lambda \left(\mu \int_0^T k(t,s) F(s,y(s))ds + g(t) \right), \qquad t \in [0,T],$$

with $\lambda \in (0,1)$. Then $y(t) = \lambda(\mu \int_0^T k(t,s)v(s)ds + g(t))$ where $v \in F(s,y)$. Consequently for $t \in [0,T]$ we have

$$\begin{aligned} |y(t)| &\le \mu k(t) \left(\sup_{[0,T]} q(t) \right) \int_0^T f(|y(s)|)ds + |g(t)| \\ &\le \mu \left(\sup_{[0,T]} k(t) \right) \left(\sup_{[0,T]} q(t) \right) f_1(|y|_0) + \sup_{[0,T]} |g(t)|. \end{aligned}$$

Thus

$$\frac{|y|_0}{\sup_{[0,T]}|g(t)| + \mu f_1(|y|_0)(\sup_{[0,T]} k(t))(\sup_{[0,T]} q(t))} \leq 1. \tag{4.10}$$

Suppose there exists $\lambda \in (0,1)$ with $|y|_0 = M_0$. Then (4.10) implies

$$\frac{M_0}{\sup_{[0,T]}|g(t)| + \mu f_1(M_0)(\sup_{[0,T]} k(t))(\sup_{[0,T]} q(t))} \leq 1$$

which contradicts (4.8). Thus any solution y to $(4.9)_\lambda$ satisfies $|y|_0 \neq M_0$. Theorem 3.6 implies that (4.1) has a solution $y \in C[0,T]$.

Case (b): Fix $\mu \leq \mu_0$. Let $M_0 > 0$ satisfy

$$\frac{M_0}{\sup_{[0,T]}|g(t)| + \mu f(M_0)(\sup_{[0,T]} k(t)) \int_0^T q(s)ds} > 1. \tag{4.11}$$

Let y be any solution of $(4.9)_\lambda$. Then for $t \in [0,T]$ we have

$$\begin{aligned} |y(t)| &\leq \mu k(t) \int_0^T q(s) f(|y(s)|) ds + |g(t)| \\ &\leq \mu k(t) f(|y|_0) \int_0^T q(s) ds + \sup_{[0,T]} |g(t)| \\ &\leq \mu f(|y|_0) \left(\sup_{[0,T]} k(t) \right) \int_0^T q(s) ds + \sup_{[0,T]} |g(t)| \end{aligned}$$

and so

$$\frac{|y|_0}{\sup_{[0,T]}|g(t)| + \mu f(|y|_0)(\sup_{[0,T]} k(t)) \int_0^T q(s)ds} \leq 1. \tag{4.12}$$

Suppose there exists $\lambda \in (0,1)$ with $|y|_0 = M_0$. Then (4.12) implies

$$\frac{M_0}{\sup_{[0,T]}|g(t)| + \mu f(M_0)(\sup_{[0,T]} k(t)) \int_0^T q(s)ds} \leq 1$$

which contradicts (4.11). □

Remarks. (i) Notice in the proof of Theorem 4.1 we only showed that any solution of $(4.9)_\lambda$ satisfies $|y|_0 \neq M_0$. We do **not** claim (and indeed it is not true in general) that any solution of $(4.9)_\lambda$ satisfies the inequality $|y|_0 \leq M_0$.

(ii) There is an obvious analogue of Theorem 4.1 if (4.3) and (4.4) are replaced by

$$\begin{cases} \text{there exists } p > 1 \text{ with } q \in L^p[0,T]. \text{ Also there exists a continuous} \\ \text{function } f_1 : [0,\infty) \to [0,\infty) \text{ such that } f_1(u) > 0 \text{ for } u > 0 \text{ and} \\ (\int_0^T f^r(|u(s)|)ds)^{\frac{1}{r}} \leq f_1(|u|_0) \text{ for any } u \in C[0,T]; \text{ here } \frac{1}{p} + \frac{1}{r} = 1. \end{cases}$$

Essentially the same reasoning as in Theorem 4.1, except we use Theorem 3.7 instead of Theorem 3.6, establishes the following result.

Theorem 4.2. *Let $F : [0,T] \times \mathbf{R} \to \mathbf{R}$ be a multivalued function with nonempty, compact values. Suppose* (2.4), (3.3), (3.4), (3.11) *and* (4.2) *are satisfied. Also assume F satisfies either* (2.1) *or* (2.2).

Case (a): *Suppose* (4.3) *and* (4.4) *hold. Let μ_0 satisfy* (4.5). *If $0 \leq \mu \leq \mu_0$, then* (4.1) *has a solution.*

Case (b): *Suppose* (4.6) *is satisfied. Let* μ_0 *satisfy* (4.7). *If* $0 \le \mu \le \mu_0$, *then* (4.1) *has a solution.*

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