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# LIPSCOMB'S UNIVERSAL SPACE IS THE ATTRACTOR OF AN INFINITE ITERATED FUNCTION SYSTEM

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ABSTRACT. Lipscomb's one-dimensional space $L(A)$ on an arbitrary index set $A$ is injected into the Tychonoff cube $I^A$. The image of $L(A)$ is shown to be the attractor of an iterated function system indexed by $A$. This system is conjugate, under an injection, with a set of right-shift operators on Baire's space $N(A)$ regarded as a code space. This view of $L(A)$ extends the fractal nature of $L(A)$ initiated in a 1992 joint paper by the author and S. Lipscomb. In addition, we give a new proof that as a subspace of Hilbert's space $l^2(A)$, the space $L(A)$ is complete and hence is closed in $l^2(A)$.

## 1. Introduction

In [5], Lipscomb introduced a universal one-dimensional metric space $L(A)$ by generalizing the standard construction of a non-decreasing map $f$ of the Cantor middle-third set $C$ onto the interval by identifying "endpoints". That map may be obtained by representing the unit interval in "ternary decimal" notation as all strings $x = .x_1x_2\cdots$ where $x_i \in \{0,1,2\}$. Then $C = \{\, x \mid x_i \in \{0,2\} \text{ for all } i \,\}$ and $f$ may be described informally by the phrase "change all the 2's to 1's and read in binary notation". Alternately, one can describe $f(C)$ as the quotient by the equivalence relation that identifies each "right-hand endpoint $.x_1\cdots x_{k-2}0222\cdots$ with the next point $.x_1\cdots x_{k-2}2000\cdots$ in $C$. Lipscomb's construction $L(A)$ replaces the set $\{0,2\}$ with an arbitrary discrete set $A$. Formally, let $N(A) = \prod_{i=1}^{\infty} A_i$ with the product topology. Then $N(A)$ is called Baire's space on $A$ and $L(A)$ is the (at most) 2-to-1 quotient of $N(A)$, just as the unit interval $I$ is the (at most) 2-to-1 quotient of $C$.

In [1] Lipscomb's space $L(A)$ on an arbitrary index set $A$ was imbedded in Hilbert's space $l^2(A)$. The imbedding solved the problem: Find a metric for $L(A)$. The solution, of course, is the metric inherited from $l^2(A)$. There was a bonus to this solution. The imbedding provided a picture of a geometry for $L(A)$, namely, a geometry analogous to that of the Sierpiński triangle $\omega^2$ in $E^2$ and Mandlebrot's fractal skewed web $\omega^3$ in $E^3$ [2, p.142]. In particular, picturing Hilbert's space $l^2(\{1,2\})$ as the plane $E^2$, we see the imbedded image of $L(\{0,1,2\})$ as $\omega^2 \subset \Delta^2$, where the 2-simplex $\Delta^2$ has its vertices at $(0,0),(1,0),(0,1) \in E^2$. And picturing

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$l^2(\{1,2,3\})$ as 3-space $E^3$, we see the imbedded image of $L(\{0,1,2,3\})$ as $\omega^3 \subset \Delta^3$, where the 3-simplex $\Delta^3$ has vertices at the origin and the terminal points of the three orthonormal basis vectors. And for $A = \{0,1,2,\ldots,n\}$, we picture $l^2(\{1,2,\ldots,n\})$ as $E^n$ and the $n$-web $\omega^n \subset \Delta^n$ as the homeomorph of $L(A)$ [1].

Recall [3] that the Sierpiński triangle $\omega^2 \subset E^2$ can be viewed as the attractor of an iterated function system (IFS) $W = \{w_0, w_1, w_3\}$ of affine transformations. Indeed, each $\omega^n \subset E^n$ is the attractor of an IFS $W = \{w_0, w_1, \ldots, w_n\}$ of $n+1$ affine transformations.

Turning to the infinite case, we shall show that the situation is entirely analogous to the finite case. At first, however, we observe that the imbedded version $\omega^A$ of $L(A)$ is a subset of a standard $|A|$-simplex $\Delta^A$, which is both a *subspace* of $l^2(A)$ and a *subset* of Tychonoff's cube $I^A$. (Recall that Tychonoff's cube $I^A$, having the product topology, is a compact subspace of generalized Euclidean space $E^A$.) To avoid confusion, let $\omega^A$ denote the imbedded version of $L(A)$, i.e., $\omega^A$ has the $l^2(A)$-induced topology. Let $\omega_c^A$ denote the space whose underlying set is that of $\omega^A$ but whose topology is induced from the Tychonoff cube $I^A$. In this paper, we shall show that $\omega_c^A$ is the attractor of an IFS $W = \{w_a\}_{a\in A}$ containing affine transformations of $E^A$. It is an open problem to construct $\omega^A$ as the attractor of an IFS containing affine transformations of $l^2(A)$.

In addition to proving that $\omega_c^A$ is an attractor we also prove that $\omega^A$ is complete, making $L(A)$ topologically complete.[1]

## 2. Background and notation

We follow the notation of [1] and single out a point $z$ of $A$, defining $A' = A - \{z\}$. (The notation $z$ is a mnemonic for *z*ero.) The points of $l^2(A)$ are collections of real numbers indexed by points of $A'$. Thus if $E$ is the set of real numbers, then $x \in l^2(A)$ means $x = \{x_a\} \in E^{A'}$ such that $x_a = 0$ for all but countably many $a \in A'$ and $\sum x_a^2$ converges. The topology of $l^2(A)$ is induced from the metric $d(x,y) = \sqrt{\sum_a (x_a - y_a)^2}$. We think of $x_a$ as the $a$th coordinate of $x$. Which coordinates happen to be non-zero will, of course, vary from point to point of $l^2(A)$. Similarly, since $A$ will be infinite, we define

$$I^A \equiv \prod_{a\in A'} I_a,$$

where $I^A$ is the Tychonoff cube, i.e., the product space of copies $I_a$ of the closed unit interval $I = [0,1]$, indexed by points of $A'$.

From [1], which laid the foundation for our present discussion, we also let $A$ denote a discrete space and

$$N(A) = \prod_{n=1}^{\infty} A_n \quad \text{(each } A_n \text{ is a copy of } A\text{)},$$

the topological product of countably many copies $A_n$ of $A$. The space $N(A)$ is usually known as *Baire's space* but here we shall also consider $N(A)$ as a generalization of *code space* [3] where $A$ is viewed as the symbol set.

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[1] In a private communication to S. Lipscomb, Professor Ivan Ivanšić of the University of Zagreb indicated that (his Ph. D. student) U. Milutinović [4] has proved the topological completeness of $L(A)$.

The points of $N(A)$ consist of all sequences $\boldsymbol{a} = a_1 a_2 \cdots a_n \cdots$ with $a_n \in A$. The space $N(A)$ is metric — define

$$d(\boldsymbol{a}, \boldsymbol{a}') = \begin{cases} \frac{1}{k} & \text{if } k \text{ is the first index where } a_i \neq a_i', \\ 0 & \text{if } \boldsymbol{a} = \boldsymbol{a}'. \end{cases}$$

Lipscomb's space $L(A)$ is obtained via a projection or identification map

$$p : N(A) \to L(A).$$

If $|A| = 2$, then the map $p$ is the identification of adjacent endpoints in a Cantor space $N(A)$ and $L(A)$ is a closed interval. This case is similar to the case where $A$ is arbitrary. Indeed, for any $A$, each point of $L(A)$, being an equivalence class in the partition induced by $p$, contains at most two members. For instance, $p(aa\cdots) = \{aa\cdots\}$ is a singleton set. If a point $a_1 a_2 \cdots \in N(A)$ has an infinite number of distinct $a_i$, then $p(a_1 a_2 \cdots)$ is also a singleton set. The points $a_1 a_2 \cdots a_{k-1} a_k a_k \cdots$ that are eventually constant sequences are called *endpoints* of $N(A)$. Moreover, only eventually constant sequences $a_1 a_2 \cdots$ that contain at least two distinct $a_i$ are identified with other points. The quotient map $p$ identifies adjacent endpoints, i.e., $p$ identifies $a_1 \cdots a_{k-2} a_{k-1} \overline{a_k} = a_1 \cdots a_{k-2} a_{k-1} a_k a_k \cdots$ with the point $a_1 \cdots a_{k-2} a_k \overline{a_{k-1}}$. A doubleton set $\alpha \in L(A)$ is called *rational* while all others are called *irrational*. Any member $\boldsymbol{a} = a_1 a_2 \cdots$ of $\alpha \in L(A)$ is an *expansion of* $\alpha$.

When $A = \{0, 1\}$, the space $L(A)$ is the unit interval and each expansion $a_1 a_2 \cdots$ of a point in $L(A)$ yields a binary representation "$.a_1 a_2 \cdots$," i.e., $p(a_1 a_2 \cdots) = .a_1 a_2 \cdots$. In the general case where $A$ is arbitrary and $a_1 a_2 \cdots$ is an expansion of $\alpha \in L(A)$, then *the A-representation* of $\alpha$ is "$.a_1 a_2 \cdots$ ." That is,

$$p(a_1 a_2 \cdots) = \alpha = .a_1 a_2 \cdots$$

where the point preceding $a_1$ is called the *A-point*. (This notation is due to S. Lipscomb.)

Turning to the imbedding map $f : L(A) \to l^2(A)$ introduced in [1], let $b \in A'$, let $\boldsymbol{a} = a_1 a_2 \cdots \in N(A)$, and let

$$K_b(\boldsymbol{a}) = \{\, k \mid a_k = b\}.$$

Then for $.a_1 a_2 \cdots = \alpha \in L(A)$, define

$$\alpha_b = \begin{cases} \sum_{k \in K_b(\boldsymbol{a})} \frac{1}{2^k} & \text{if } K_b(\boldsymbol{a}) \neq \emptyset, \\ 0 & \text{if } K_b(\boldsymbol{a}) = \emptyset \end{cases}$$

and let $f : L(A) \to l^2(A)$ be given by

$$f(\alpha) = \{\alpha_b\} \in l^2(A).$$

Note that $f$ takes the point $p(\boldsymbol{z}) = \{zzz\cdots\} = .zz\cdots$ whose expansion is the constant sequence $zz\cdots$ to the zero $\mathbf{0}$ of $l^2(A)$. For any other $b \in A'$, the imbedding $f$ sends the point $.bb\cdots$ to the unit vector $\boldsymbol{u}_b \in l^2(A)$, where

$$a\text{th coordinate of } \boldsymbol{u}_b = \begin{cases} 0 & \text{if } a \neq b, \\ 1 & \text{if } a = b. \end{cases}$$

In short, for points in $L(A)$ having a constant sequence expansion, $f$ sets up the following one-one correspondence:

$$\begin{aligned}\{zz\cdots\} &\mapsto \mathbf{0}\\ \{bb\cdots\} &\mapsto \boldsymbol{u}_b, \quad \text{for } b\in A'.\end{aligned}$$

From this correspondence it is easy to envision the action of $f$ on $L(A)$. For example, if $\alpha = .a_1a_2\cdots \in L(A)$ where $\{a_1, a_2, \dots\} = \{a, b\} \subset A'$, then there is a unique real number $t$, $0 \le t \le 1$, such that

$$f(\alpha) = t\boldsymbol{u}_a + (1-t)\boldsymbol{u}_b.$$

Furthermore, for each triple subset $\{a, b, c\}$ of $A'$, the image $\omega^A$ of $f$ meets the 3-simplex $\Delta^3 = [\boldsymbol{u}_a, \boldsymbol{u}_b, \boldsymbol{u}_c]$ in the (Mandlebrot's) fractal skewed web whose vertices are those of $\Delta^3$.

More generally, any $x \in l^2(A)$ is a countable linear combination

$$x = \textstyle\sum_{A'} x_b\boldsymbol{u}_b.$$

That is, viewing $x$ and each $\boldsymbol{u}_b$ as function $A' \to E$,

$$x(a) = \textstyle\sum_{A'} x_b\boldsymbol{u}_b(a) = x_a\boldsymbol{u}_a(a) = x_a.$$

It is therefore natural to define the *standard* $|A|$-*simplex* $\Delta^A$ in $l^2(A)$ as

$$\Delta^A = \{\, x \in l^2(A) \mid \text{ each } x_b \ge 0;\ \textstyle\sum_{A'} x_b \le 1\,\}.$$

Thus, the vertices of $\Delta^A$ consist of $\mathbf{0}$ and each $\boldsymbol{u}_b$ for $b \in A'$. If we restrict the non-zero coordinates of $x \in l^2(A)$ to be finite in number, then we get every $n$-simplex $\Delta^n$ as a face of $\Delta^A$. In addition, from [1, Corollary 8], $f$ maps each $\alpha \in L(A)$ into $\Delta^A$. We will define

$$f\big(L(A)\big) = \omega^A \subset \Delta^A.$$

As a point set we have

$$\Delta^A \subset I^A,$$

where the topology of $\Delta^A$, being induced from $l^2(A)$, differs from that induced from $I^A$.

Turning to $I^A$, for any $x = \{x_a\}$ in $I^A$, let $\frac{1}{2}x = \big\{\frac{1}{2}x_a\big\} \in I^A$ and define

$$\tfrac{1}{2}X = \{\, \tfrac{1}{2}x \,|\, x \in X\,\}.$$

In addition, for each real number $r$ such that $0 \le r \le 1$, let

$$x + r_a = x + r\boldsymbol{u}_a = y \in I^A, \quad \text{where } \ y_b = \begin{cases} x_b & \text{if } b \ne a,\\ \min\{1, x_a + r\} & \text{if } b = a.\end{cases}$$

More generally, for any $X \subset I^A$ and any real number $r$, let

$$X + r_a = X + r\boldsymbol{u}_a = \{\, x \,:\, x = x' + r_a,\ x' \in X\} \subset I^A.$$

Finally, a subbasic open set $\langle G_a\rangle$ in $I^A$ is a product of sets, the $a$th factor of which is an open subset $G_a$ of $I$, while for $b \ne a$, the $b$th factor is $I_b$. In terms of this subbasis, the basic open sets are of the form $G = \langle G_{a_1}, G_{a_2}, \dots, G_{a_n}\rangle = \bigcap_{i=1}^n \langle G_{a_i}\rangle$.

## 3. Preliminary observations

We begin with several lemmas.

**Lemma 1.** *The map* $\phi(x) = \frac{1}{2}x$ *on* $I^A$ *is continuous.*

*Proof.* For each $a \in A'$, let $\phi_a : I_a \to I_a$ be the continuous map given by $x \mapsto \frac{1}{2}x$. Then observe that $\phi = \times_{A'}\phi_a$ and is a product map of continuous maps and therefore continuous. □

**Lemma 2.** *For each real number* $r$, $0 \leq r \leq 1$, *and each* $a \in A'$, *the map* $\theta(x) = x + r_a$ *is continuous on* $I^A$.

*Proof.* Let $a_0 \in A'$ be fixed. Then for each $a \in A' - \{a_0\}$, let $\theta_a : I_a \to I_a$ be the continuous identity map $x \mapsto x$. And for $a = a_0$, let $\theta_a : I_a \to I_a$ be the continuous map given by $x \mapsto min\{1, x + r\}$, which is just a shift by $r$, followed by a retraction to 1 if necessary to stay in $[0, 1]$. Then observe that $\theta = \times_{A'}\theta_a$ is a product map of continuous maps and therefore continuous. □

**Lemma 3.** *Let* $G$ *be open in* $I^A$. *Then* $G \cap \omega^A$ *is open in* $\omega^A$.

*Proof.* Let $G = \langle G_a \rangle$ be a subbasic open set. Let $y \in G \cap \omega^A$. Then $y_a \in G_a$. There exists a neighborhood $N(y_a, \varepsilon) \subset G_a$ for some $\varepsilon > 0$. Let $w \in N(y, \varepsilon/2)$ in $\omega^A$. Then $w \in G$. Otherwise $w_a \notin N(y_a, \varepsilon)$. This implies $(w_a - y_a) \geq \varepsilon$. Therefore $d(y, w) \geq \varepsilon$ in $\omega^A$, contradicting the selection of $w$. Therefore $G \cap \omega^A$ is the union of neighborhoods about each point in $G \cap \omega^A$. Therefore $G \cap \omega^A$ is open in $\omega^A$. □

This lemma shows that the injection map $\omega^A \hookrightarrow I^A$ is continuous. Define $\omega_c^A$ as the image of the injection of $\omega^A$ into $I^A$. As a result, we shall use "$f$" to denote both the homeomorphism $f : L(A) \to \omega^A$ and the continuous injection $f : L(A) \to \omega_c^A \subset I^A$, the particular choice will be clear from the context.

These lemmas immediately yield the following lemma.

**Lemma 4.** *Let* $X$ *be compact in* $I^A$, *let* $a \in A'$, *and let* $r$ *be a real number,* $0 \leq r \leq 1$. *Then both sets* $X + r_a$ *and* $\frac{1}{2}X$ *are compact.*

**Theorem 5.** *Let* $C$ *be a compact subset of* $I^A$, *let* $B \subset A$, *and let* $r : B \to [0, 1]$ *be any map. If for each* $b \in B$, $C_b = C + r(b)\boldsymbol{u}_b = C + r_b$, *then* $K = C \cup (\bigcup_B C_b)$ *is compact.*

*Proof.* First, observe that the previous lemma shows that each $C_b$ is compact. So, let $\mathcal{G}$ be a covering of $K$ with basic open sets. Then, since $C$ is compact, a finite subset $\mathcal{F} \subset \mathcal{G}$ covers $C$, i.e., $C \subset V = \bigcup_{\mathcal{F}} G$. For each $G \in \mathcal{F}$ a finite set $A_G \subset A'$ exists such that $G$ is the intersection of the subbasic sets $\langle G_a \rangle$, $a \in A_G$. (In other words, $x \in G$ if, and only if, $x_a \in G_a$ for each $a \in A_G$.) Then, since $\bigcup_{\mathcal{F}} A_G = F$ is finite and each $C_b$ $(b \in F)$ is compact, $\bigcup_F C_b$ is compact. It therefore follows that we can finish the proof by showing that $b \in B - F$ implies $C_b \subset V$. So, for such a $b$, let $x \in C_b$. Then $x = x' + r_b$, where $x' \in C$. Then $C \subset V$ implies $x' \in V$. Consequently, $G \in \mathcal{F}$ exists such that $x' \in G$. Or, in other words, ${x_a}' \in G_a$ for each $a \in A_G$. Then, since $b \notin A_G$, $x_a = {x_a}' \in G$ for each $a \in A_G$ implying $x \in G$. Since $G \subset V$, $x \in V$, which finishes the proof. □

## 4. An infinite hyperbolic iterated function system (IFS)

For each $a \in A$, define a (clearly continuous) *shift map* $S'_a : N(A) \to N(A)$ by $a_1a_2\cdots \mapsto aa_1a_2\cdots$. Each such map has its counterpart $S_a : L(A) \to L(A)$ given by $.a_1a_2\cdots \mapsto .aa_1a_2\cdots$. It is obvious that $S_a$ is well defined. Moreover, since

$$p \circ S'_a(a_1a_2\cdots) = p(aa_1a_2\cdots) = .aa_1a_2\cdots \quad \text{and}$$
$$S_a \circ p(a_1a_2\cdots) = S_a(.a_1a_2\cdots) = .aa_1a_2\cdots$$

we have $S_a \circ p = p \circ S'_a$, i.e., the top square in diagram (1) is commutative.

$$\begin{array}{ccc} N(A) & \xrightarrow{S'_a} & N(A) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ L(A) & \xrightarrow{S'_a} & L(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ I^A & \xrightarrow{w_a} & I^A. \end{array} \tag{1}$$

It follows, then, that $S_a$ is continuous, i.e., since $p$ is a quotient map and $S_a \circ p$ $(= p \circ S'_a)$ is continuous, $S_a$ is continuous.

To make the whole diagram (1) commutative, we define $w_a : I^A \to I^A$ that is conjugate with $S_a$ under $f$ ($w_af = fS_a$). More precisely, choose $a \in A$ and let

$$w_a(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2}\boldsymbol{u}_a & \text{if } a \neq z, \\ \frac{1}{2}x & \text{if } a = z. \end{cases}$$

We note that the collection $\{w_a\}_A$ is a natural generalization of the well-known (finite) IFS used to generate a Sierpiński triangle in the plane with vertices at $(0,0)$, $(0,1)$, and $(1,0)$ (see [3]).

To see the commutativity $w_af = fS_a$, we begin with $a \neq z$. We compare $f(.a_1a_2\cdots)$ with $f(.aa_1a_2\cdots)$ coordinate-wise. First, consider a coordinate $b \neq a$. Then by the definition of $f$, the $b$th coordinate of $f(.aa_1a_2\cdots)$ is $\frac{1}{2}$ that of $f(.a_1a_2\cdots)$, showing that $w_a$ does indeed match these $b$th coordinates. Second, comparing $a$th coordinates, that of $f(.aa_1a_2\cdots)$ is obtained from that of $f(.a_1a_2\cdots)$ by multiplying by $\frac{1}{2}$ and then adding $\frac{1}{2}$. Thus, in either case, $w_af = fS_a$ when $a \neq z$. The argument that $w_zf = fS_z$ is similar.

**Lemma 6.** *If $x = f(.a_1a_2\cdots) \in \omega_c^A$, then for any $n \geq 1$, $x \in w_{a_1}\ldots w_{a_n}I^A$.*

*Proof.* We use $w_af = fS_a$. First, note that

$$.a_1a_2\cdots = S_{a_1}\cdots S_{a_n}(.a_{n+1}a_{n+2}\cdots).$$

Then calculate:

$$\begin{aligned} x &= f(S_{a_1}\cdots S_{a_n}.a_{n+1}\cdots) = w_{a_1}f(S_{a_2}\cdots S_{a_n}.a_{n+1}\cdots) \\ &= w_{a_1}w_{a_2}f(S_{a_3}\cdots S_{a_n}.a_{n+1}\cdots) = \cdots \\ &= w_{a_1}\cdots w_{a_n}f(.a_{n+1}\cdots) \in w_{a_1}\cdots w_{a_n}I^A. \qquad \square \end{aligned}$$

## 5. The compact space $\omega_c^A$ is an attractor for $W = \{w_a\}_A$

We need some preliminary constructions and definitions.

**Definition 7.** For $X \subset I^A$ we say that $X$ is invariant under $W$ whenever

$$X = W(X) = \bigcup_{a \in A} w_a(X).$$

We inductively construct a nested sequence $\{I_n\}_{n=0}^{\infty}$ of compact sets. Let $I_0 = I^A$ and $I_1 = W(I_0)$. Then $I_1 \subset I_0$ and both $I_0$ and $I_1$ are compact — $I_1$ is compact by Theorem 5. Recursively, define $I_{n+1} = W(I_n)$. Assuming that $I_n \subset I_{n-1}$, the calculation

$$I_{n+1} = W(I_n) \subset W(I_{n-1}) = I_n$$

shows that $I_{n+1} \subset I_n$ for every $n \geq 0$. Moreover, another application of Theorem 5 shows that the compactness of $I_n$ implies that of $I_{n+1}$. With $\{I_n\}_{n=0}^{\infty}$ defined we let $P = \bigcap_{n=0}^{\infty} I_n$, which is obviously also compact. (Note that $\mathbf{0} \in P \neq \emptyset$ since $z \in A$ and $w_z(\mathbf{0}) = \mathbf{0}$.)

**Lemma 8.** *We have $\omega_c^A \subset P$.*

*Proof.* First, observe that each

$$I_n = \bigcup_{a,b,\ldots,q \in A} \underbrace{w_a w_b \cdots w_q}_{n} I^A.$$

Now let $x = f(.a_1 a_2 \cdots) \in \omega_c^A$. Then by Lemma 6, $x \in w_{a_1} \ldots w_{a_n} I^A \subset I_n$ for all $n$, showing $x \in P$. □

To see that $P \subset \omega_c^A$, we let $x \in P$ and show that an $\boldsymbol{a} = a_1 a_2 \cdots \in N(A)$ exists such that

$$\begin{aligned} & x \in S_1 = w_{a_1} I^A \\ & x \in S_2 = w_{a_1} w_{a_2} I^A \qquad (2) \\ & \vdots \end{aligned}$$

First, however, we need the following technical lemma.

**Lemma 9.** *Let $a \in A$, $y \in I^A$, and $x = w_{a_1} \ldots w_{a_n} y$. Then we have the following:*
(*i*) *If $a_1 = a \neq z$, then $x_a \geq \frac{1}{2}$.*
(*ii*) *If $j$ is the smallest index such that $a_j \neq a$, then*

$$x_a \leq \tfrac{2^j - 1}{2^j} < 1.$$

(*iii*) *If $a_1 \neq a$, then $x_a \leq \frac{1}{2}$.*
(*iv*) *If $a_1 \neq a$ and $x_a \geq \frac{1}{2}$, then $a_2 = \cdots = a_n = a$.*

*Proof.* (*i*) If $a_1 = a \neq z$, then for $v = w_{a_2} \cdots w_{a_n} y$, it follows that $0 \leq v_a \leq 1$ and therefore $x_a = \frac{1}{2} v_a + \frac{1}{2} \geq \frac{1}{2}$. (*ii*) First suppose $a_1 \neq a$. Then $x_a \leq \frac{1}{2} = \frac{2^1 - 1}{2^1}$ since the $a$th component of $w_{a_2} \cdots w_{a_n} y \leq 1$. Second, suppose $a_1 = a$. Then $j \geq 2$ and, letting $v = w_{a_j} \ldots w_{a_n} y$, we have $x = w_{a_1} \cdots w_{a_{j-1}} v$. From the previous case the maximum possible value of $v_a$ is $\frac{1}{2}$. In addition, the maximum possible value of $x_a$ occurs whenever $a_1 = \cdots = a_{j-1} = a$. Thus,

$$x_a \leq \tfrac{2^j - 1}{2^j} < 1.$$

(*iii*) This is statement (*ii*) where $j = 1$. (*iv*) If a smallest index $j \geq 2$ exists such that $a_j \neq a$, then for $v = w_{a_2} \cdots w_{a_n} y$, it follows from (*ii*) that

$$v_a \leq \tfrac{2^{j-1}-1}{2^{j-1}} \quad \text{which implies} \quad x_a = \tfrac{1}{2} v_a = \tfrac{2^{j-1}-1}{2^j} < \tfrac{1}{2},$$

which is a contradiction. □

**Theorem 10.** *The space $\omega_c^A = P$ and is therefore compact.*

*Proof.* By Lemma 6, it suffices to show $P \subset \omega_c^A$. To do this, let $x \in P$. Then, by construction of $P$,

$$x \in w_{a_{11}} I^A = T_1$$
$$x \in w_{a_{21}} w_{a_{22}} I^A = T_2$$
$$\vdots$$

The trouble is we can't say anything about the limit of the initial strings of $w$'s, if there is one. But from $\{T_n\}$, we need to extract a sequence $\{S_n\}$ as defined in (2). The inductive construction of $\{S_n\}$ depends on the initial strings of $w$'s that define $\{T_n\}$. First, we select $w_{a_1}$. There are three cases to consider:

(a) A $w_a$ exists that initiates infinitely many $T_n$'s.
(b) At least two $w$'s, say $w_a$ and $w_b$, exist that initiate infinitely many $T_n$'s.
(c) No initial $w$ initiates infinitely many $T_n$'s.

If (a), then we define $w_{a_1} = w_a$ and $w_{a_1} I^A = S_1$. Thus, $x \in w_{a_1} I^A = S_1$. If (b), then we can assume that $a \neq z$ and that $w_a \neq w_b$. In this case, for each $n \geq 1$ we have

$$x \in w_a w_{p_1} \dots w_{p_n} I^A, \tag{3}$$
$$x \in w_b w_{q_1} \dots w_{q_n} I^A. \tag{4}$$

From (*i*) of Lemma 9 and (3), $x_a \geq 1/2$. And from (*iv*) of Lemma 9 and (4), $q_1 = \cdots = q_n = a$. In this case, then, we can construct all $S_n$ because

$$x \in w_b \underbrace{w_a \cdots w_a}_{n} I^A \quad \text{for every } n > 1.$$

That is, for each $n \geq 2$, define

$$S_n = w_b \underbrace{w_a \cdots w_a}_{n-1} I^A.$$

Finally, we show (c) is impossible. For otherwise, select $a \neq z \neq b \neq a$ such that

$$x \in w_a I^A, \tag{5}$$
$$x \in w_b I^A. \tag{6}$$

Furthermore, select $c$ such that $a \neq c \neq b$, with

$$x \in w_c w_{p_2} \dots w_{p_n} I^A, \quad n \geq 2. \tag{7}$$

Again, (*i*) of Lemma 9 and (5) show that $x_a \geq \frac{1}{2}$ while (*iv*) of Lemma 9 and (7) show that $p_2 = \cdots = p_n = a$. But a similar argument concerning $x_b$ shows that $p_2 = \cdots = p_n = b$, which contradicts $a \neq b$. In short, case (c) is impossible while case (b) provides the whole sequence $\{S_n\}$. So the inductive step only need concern case (a), i.e., suppose for some fixed $k \geq 1$

$$x \in w_{a_1} \dots w_{a_k} I^A$$

where $w_{a_1} \dots w_{a_k}$ initiates infinitely many $T_n$. Since each $w_{a_i}$ is one-one, a $y \in I^A$ must exist where $x = w_{a_1} \dots w_{a_k} y$. As a result $y$ must also be contained in every $T_n$ having $w_{a_1} \cdots w_{a_k}$ as initial segment. To define $a_{k+1}$, then, we can repeat the argument for finding $a_1$, this time however, we use $y$ in place of $x$ and we replace $\{T_n\}$ with $\{T'_m\}$, which we generate by "chopping off" the first $k$ $w$'s from those $T_n$ with initial segment $w_{a_1} \dots w_{a_k}$. At any rate we find $\boldsymbol{a} = a_1 a_2 \cdots \in N(A)$ such that

$$\begin{aligned} &x \in w_{a_1} I^A \subset I_1 \\ &x \in w_{a_1} w_{a_2} I^A \subset I_2 \\ &\vdots\ . \end{aligned}$$

Now consider $x' = f(.a_1 a_2 \cdots) \in \omega_c^A$. From Lemma 6, we then have

$$x' \in w_{a_1} \dots w_{a_n} I^A \quad \text{for all } n \geq 1.$$

Thus, both $x$ and $x' \in P$. We show $x = x'$. Otherwise, for some $a \neq z$, we have $x_a - x'_a > \varepsilon > 0$. Let $N$ be such that $\frac{1}{2^N} < \varepsilon$. Then both

$$x \text{ and } x' \in w_{a_1} \dots w_{a_N} I^A = \textstyle\prod_{a \in A'} X_a$$

where each $X_a$ is a closed interval of length $\frac{1}{2^N}$, showing $|x_a - x'_a| < \varepsilon$. Thus, $x = x' \in \omega_c^A$. □

**Theorem 11.** *The compact set $\omega_c^A$ is invariant under $W$.*

*Proof.* We show $\omega_c^A = W\left(\omega_c^A\right) = \bigcup_{a \in A} w_a\left(\omega_c^A\right)$. Let $x = f(.a_1 a_2 \cdots) \in \omega_c^A$. Then $f(.a_2 a_3 \cdots) \in \omega_c^A$, $a_1 \in A$, and from diagram (1),

$$x = f(.a_1 a_2 \cdots) = f S_{a_1}(.a_2 a_3 \cdots) = w_{a_1}\left(f(.a_2 a_3 \cdots)\right) \in \bigcup_{a \in A} w_a(\omega_c^A) = W\left(\omega_c^A\right).$$

Therefore $\omega_c^A \subset W\left(\omega_c^A\right)$. To see the reverse inclusion, let $x \in W\left(\omega_c^A\right)$. Then for some $.a_2 a_3 \cdots \in L(A)$ and $a_1 \in A$, we have

$$x = w_{a_1} f(.a_2 a_3 \cdots) = f\left(S_{a_1} .a_2 a_3 \cdots\right) = f(.a_1 a_2 \cdots) \in \omega_c^A. \qquad \square$$

**Theorem 12.** *The set $\omega_c^A$ is the only $W$-invariant compact subset of $I^A$.*

*Proof.* We show no other closed (= compact) subset $B$ of $I^A$ is invariant under $W$. Consider first the case where $B \not\subset \omega_c^A = P = \bigcap_n I_n$. Then $B$ fails to lie inside of some $I_n$. But by construction of the $I_n$'s, if $B$ were invariant, then $W(B) \subset I_n$ for every $n$. Therefore $W(B) \neq B$, showing that $B$ is not invariant under $W$. For the other case, where $B$ is a proper subset of $\omega_c^A$, let

$$B' = (f \circ p)^{-1}(B) \subset N(A).$$

Then, since $B'$ is a proper closed subset of $N(A)$, a point $\boldsymbol{a} = a_1 a_2 \cdots \in N(A) - B'$ exists and we can consider one of its neighborhoods $V = \langle a_1, \dots, a_n \rangle$ where $V \cap B' =$

$\emptyset$. Then for $\boldsymbol{b} \in B'$, since $S'_{a_1} \cdots S'_{a_n}\boldsymbol{b} \in V$, we have $S'_{a_1} \cdots S'_{a_n}\boldsymbol{b} \notin B'$. Thus, via conjugacy,

$$(f \circ p)\boldsymbol{b} = x \in B, \qquad \text{but} \qquad \begin{aligned} &(f \circ p)S'_{a_1} \cdots S'_{a_n}\boldsymbol{b} \notin B \\ &\Rightarrow w_{a_1} \cdots w_{a_n}(f \circ p)\boldsymbol{b} \notin B \\ &\Rightarrow w_{a_1} \cdots w_{a_n}(x) \notin B, \end{aligned}$$

which contradicts the invariance of $B$. Having exhausted all cases, the proof is complete. □

## 6. Topological completeness of $L(A)$

Convergence in $L(A)$ is ultimately convergence in $N(A)$. We therefore begin with a lemma concerning convergence in $N(A)$.

**Lemma 13.** *Let $\{\boldsymbol{a}_n\}$ be an infinite sequence in $N(A)$ with no limit point. For each index $n$, let $\boldsymbol{a}_n = a_1^n a_2^n \cdots$. Then there exists a subsequence $\{\boldsymbol{a}_m\}$ of $\{\boldsymbol{a}_n\}$ and a fixed index $i > 0$ such that $\{a_i^m\} = \{a^m\}$ is an infinite sequence in $A$.*

*Proof.* Suppose no such index exists. Then $\{a_1^n\}$ is finite and an $a_1 \in \{a_1^n\}$ exists that is the first coordinate of each member in an infinite subsequence $S_1$ of $\{\boldsymbol{a}_n\}$. Inductively, suppose $a_1 \cdots a_k$ is the initial $k$-segment of each member in an infinite subsequence $S_k$ of $\{\boldsymbol{a}_n\}$. Then since $\{\, a_{k+1}^n \,|\, \boldsymbol{a}_n \in S_k \,\}$ is finite, one of its members, say $a_{k+1}$, is the $(k+1)$st coordinate of an infinite number of members of $S_k$. As a result $a_1 \cdots a_{k+1}$ is the initial $(k+1)$-segment of each member in an infinite subsequence $S_{k+1}$ of $\{\boldsymbol{a}_n\}$. Thus, we get such a subsequence $S_k$ for every $k > 0$. It follows that the point

$$\boldsymbol{a} = a_1 a_2 \cdots \in N(A)$$

is a limit point of $\{\boldsymbol{a}_n\}$, i.e., for any $k > 0$, the $k$th coordinate of $\boldsymbol{a}$ agrees with infinitely many points of $\{\boldsymbol{a}_n\}$. □

**Theorem 14.** *Lipscomb's metric space $L(A)$ is topologically complete.*

*Proof.* It suffices to show that the homeomorphic image $\omega_A \subset l^2(A)$ is complete in the induced metric. So let $\{x^n\}$ denote an infinite Cauchy sequence in $\omega^A$. For each index $n$, choose $\boldsymbol{a}_n = a_1^n a_2^n \cdots \in N(A)$ such that

$$f \circ p(\boldsymbol{a}_n) = f(.\boldsymbol{a}_n) = x^n.$$

Suppose $\{x^n\}$ has no limit point in $\omega^A$. Then $\{.\boldsymbol{a}_n\}$ is an infinite sequence in $L(A)$ with no limit point. And in turn, since $p$ is a continuous map, $\{\boldsymbol{a}_n\}$ is an infinite sequence in $N(A)$ with no limit point. By Lemma 13, there exists a subsequence $\{\boldsymbol{a}_m\}$ of $\{\boldsymbol{a}_n\}$ and an index $i > 0$ such that $\{a_i^m\} = \{a^m\}$ is an infinite sequence in $A$. We can also suppose $z \notin \{a^m\}$. Then for every $x^m$, since $f(.\boldsymbol{a}^m) = x^m$ and the $i$th coordinate of $.\boldsymbol{a}^m$ is $a^m$,

$$x^m_{a^m} \geq 1/(2^i). \tag{8}$$

But $\{x^n\}$ is Cauchy, implying that for some $N$ and all $m, k > N$,

$$d\big(x^m, x^k\big) < 1/(2^{i+1}). \tag{9}$$

Fix a $k > N$ and from $\{\boldsymbol{a}_m\}$ select an $\boldsymbol{a}_m$ where $m > N$. Then we can show that

$$x^k_{a^m} > 1/(2^{i+1}). \tag{10}$$

To see why (10) is true, suppose otherwise. Then $x^k_{a^m} \leq 1/(2^{i+1})$ and (8) show

$$x^m_{a^m} - x^k_{a^m} \geq 1/(2^{i+1}),$$

which implies

$$d\big(x^m, x^k\big) \geq \Big(\big[x^m_{a^m} - x^k_{a^m}\big]^2\Big)^{1/2} \geq 1/(2^{i+1}),$$

which contradicts (9). Thus for large $k$, inequality (10) is true for an infinite number of $m$, contradicting $x^k \notin l^2(A)$. Thus, $\{x^n\}$ has a limit point in $\omega^A$. □

Since a complete subspace of a metric space must be a closed set, Theorem 14 shows that $\omega^A$ is closed in $\ell^2(A)$. Thus, we have the following corollary.

**Corollary 15.** *The $f$-image $\omega^A$ of $L(A)$ is a closed subset of $\ell^2(A)$.*

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