PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 8, August 1996

A NON-HOMOGENEOUS ZERO-DIMENSIONAL X SUCH THAT $X \times X$ IS A GROUP

FONS VAN ENGELEN

(Communicated by Franklin D. Tall)

ABSTRACT. We provide an example of a zero-dimensional (separable metric) absolute Borel set X which is not homogeneous, but whose square $X \times X$ admits the structure of a topological group. We also construct a zero-dimensional absolute Borel set Y such that Y is a homogeneous non-group but $Y \times Y$ is a group. This answers questions of Arhangel'skiĭ and Zhou.

1. INTRODUCTION

In [3], Arhangel'skiĭ asked if there exists a non-homogeneous (compact) space X such that $X \times X$ is homogeneous, and in [18], Zhou asked whether there exists a zero-dimensional first-countable non-group X such that $X \times X$ is a group. Examples of infinite dimension and finite positive dimension answering Arhangel'skiĭ's question affirmatively have been constructed by van Mill [16], Ancel and Singh [1], and Ancel, Duvall and Singh [2]. In this note we will answer Arhangel'skiĭ's question in the zero-dimensional case, at the same time answering Zhou's question, by constructing a non-homogeneous zero-dimensional (separable metric) absolute Borel set X such that $X \times X$ admits the structure of a topological group. We will also show that there is a *homogeneous* non-group Y such that $Y \times Y$ is a group. Finally, we will prove that our examples are best possible in the sense that they are of minimal complexity in the Borel Wadge hierarchy.

All spaces in this note will be assumed to be separable, metrizable and zero-dimensional; in fact, it will be convenient to just assume that all our spaces are subspaces of the Cantor set 2^ω. For those familiar with the inductive definition of the Borel Wadge classes in 2^ω due to Louveau [14] and the results of [5], let us mention that our space X will be $(\omega \times Y) \cup (\{\omega\} \times (2^\omega - Y))$ in $(\omega + 1) \times 2^\omega \approx 2^\omega$, where Y is the unique homogeneous element of $Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)$. Since these papers are long and technical, and much too general for the results of this note, we will give an exposition which is much more self-contained, especially where it concerns the construction of X and Y. However, in section 5 of this note, where Wadge-minimality of X and Y is proved, we do in fact presuppose knowledge of [14] and [5].

I am indebted to the referee for some helpful comments.

Received by the editors February 20, 1995.
1991 *Mathematics Subject Classification.* Primary 54H05, 54E35, 54F65; Secondary 03E15.
Key words and phrases. Zero-dimensional, Borel, Wadge hierarchy, homogeneous.

2. Preliminaries

The notation $(h\colon)A \approx B$ means that A and B are homeomorphic (witnessed by h). Recall that a space A is *homogeneous* if for all $x, y \in A$ there exists $h\colon A \approx A$ with $h(x) = y$, and *strongly homogeneous* if all non-empty clopen subsets of A are homeomorphic; a strongly homogeneous space is homogeneous. The symbols $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$, and $\mathbf{\Delta}^0_\xi$ will be used to denote the additive, multiplicative, and ambiguous, resp., Borel classes in 2^ω (where $\mathbf{\Sigma}^0_1$ consists of all open subsets).

If $A, B \subseteq 2^\omega$, then A is *Wadge-reducible to* B (notation $A \leq_w B$) if there exists a continuous $f\colon 2^\omega \to 2^\omega$ such that $A = f^{-1}[B]$; if both $A \leq_w B$ and $B \leq_w A$, then A and B are *Wadge-equivalent*, notation $A \equiv_w B$. A *Wadge class* is a class of subsets of 2^ω of the form $[A] = \{B : B \leq_w A\}$; if $\Gamma = [A]$, then A is said to *generate* the Wadge class Γ. If Γ is any class of subsets of 2^ω, then the *dual class of* Γ is $\{A : 2^\omega - A \in \Gamma\}$, denoted by $\check{\Gamma}$ or $\Gamma^\vee$; Γ is *non-self-dual* if $\Gamma \neq \check{\Gamma}$. Γ is *continuously closed* if whenever $A \in \Gamma$ and $B \leq_w A$, then $B \in \Gamma$. It follows from the so-called *Wadge Lemma* that if Γ is a non-self-dual and continuously closed class of Borel sets, then Γ is a Wadge-class which is generated by any $A \in \Gamma - \check{\Gamma}$; in other words, if $A \in \Gamma - \check{\Gamma}$, then ($B \in \Gamma - \check{\Gamma}$ if and only if $B \equiv_w A$).

For $i \in \{0, 1\}$, define

$$Q_i = \{x \in 2^\omega : \exists m : \forall n \geq m : x_n = i\},$$

and let $P = 2^\omega - (Q_0 \cup Q_1)$. Note that P is $\mathbf{\Pi}^0_2$. If $x \in P$, then x consists of blocks of zeros separated by blocks of ones; define $\phi\colon P \to 2^\omega$ by $\phi(x)_n = 0$ iff the n^{th} block of zeros in x has even length. Note that ϕ is continuous and open.

Definition 2.1. Let Γ be a class of spaces.

(a) Γ is reasonably closed if Γ is continuously closed, and for each $A \in \Gamma$, $\phi^{-1}[A] \cup Q_0 \in \Gamma$.
(b) A space A is everywhere properly Γ if for each non-empty clopen subset U of 2^ω, $U \cap A \in \Gamma - \check{\Gamma}$.

We will make frequent use of the fact that Γ is reasonably closed if and only if Γ is closed under homeomorphisms, under unions with $\mathbf{\Sigma}^0_2$-sets, and under intersections with $\mathbf{\Pi}^0_2$-sets (see [15]).

Theorem 2.2 (Steel [17])**.** *If Γ is a reasonably closed class of Borel sets, and A and B are both everywhere properly Γ, and either both first category or both Baire, then $A \approx B$.*

It is easy to see that $\mathbf{\Sigma}^0_3$ and $\mathbf{\Pi}^0_3$ are reasonably closed, and from [12] it follows that the countable infinite product of rationals $\mathbb{Q}^\omega$ is everywhere properly $\mathbf{\Pi}^0_3$ and $2^\omega - \mathbb{Q}^\omega$ is everywhere properly $\mathbf{\Sigma}^0_3$ (if $\mathbb{Q}^\omega$ is densely embedded in 2^ω). Thus, it follows from Theorem 2.2 that $\mathbb{Q}^\omega$ (resp. $2^\omega - \mathbb{Q}^\omega$) is characterized by being first category (resp. Baire) and everywhere properly $\mathbf{\Pi}^0_3$ (resp. $\mathbf{\Sigma}^0_3$).

3. The main lemma

In [7] it was shown that all homogeneous spaces in $\mathbf{\Delta}^0_3$ contain a countable (or, in just a few cases, a σ-compact) dense subset such that any relative $\mathbf{\Pi}^0_2$-set in the space which contains it is actually homeomorphic to the space. In this section we will extend this result to arbitrary homogeneous Borel sets. It will be the main result needed in the next section to show that our homogeneous example Y is not a group.

For $s = (s_0, \ldots, s_{n-1}) \in 2^{<\omega}$ we let $|s| = n = \mathrm{dom}(s)$ be the length of s, and $f(s) = s_{n-1}$ the final element of s. The length of the empty sequence $\langle\ \rangle$ is 0. We write $s < y$ for $s \in 2^{<\omega}$ and $y \in 2^{<\omega}$ or $y \in 2^\omega$ if y properly extends s. Finally, $\mathbf{0} \in 2^\omega$ has each coordinate 0.

Let $\sigma\colon 2^{<\omega} \to 2^{<\omega}$ be such that if $s < t$, then $\sigma(s) < \sigma(t)$. Then for each $x \in 2^\omega$ there exists a unique $f_\sigma(x) \in 2^\omega$ such that for each n, $\sigma(x|n) < f_\sigma(x)$, and moreover this function $f_\sigma\colon 2^\omega \to 2^\omega$ is continuous.

Lemma 3.1. *Let $[A] \subseteq \mathcal{P}(2^\omega)$ be reasonably closed. Then $A \equiv_w \phi^{-1}[A] \cup Q_0$.*

Proof. By definition of reasonably closed, $\phi^{-1}[A] \cup Q_0 \leq_w A$. Inductively, define $\sigma\colon 2^{<\omega} \to 2^{<\omega}$ by $\sigma(\langle\ \rangle) = \langle\ \rangle$, $\sigma(s^\frown 0) = \sigma(s)^\frown 001$, and $\sigma(s^\frown 1) = \sigma(s)^\frown 01$. Then for each $x \in 2^\omega$, $\phi f_\sigma(x) = x$, so $f_\sigma^{-1}[\phi^{-1}[A] \cup Q_0] = f_\sigma^{-1}\phi^{-1}[A] = A$. Thus, also $A \leq_w \phi^{-1}[A] \cup Q_0$. □

Lemma 3.2. *Let $[A] \subseteq \mathcal{P}(2^\omega)$ be reasonably closed, and let G be a $\mathbf{\Pi}^0_2$-set in 2^ω. If $G \supseteq Q_0$, then $G \cap (\phi^{-1}[A] \cup Q_0) \equiv_w A$.*

Proof. By the previous lemma, $\phi^{-1}[A] \cup Q_0 \leq_w A$, so since Γ is closed under intersections with $\mathbf{\Pi}^0_2$-sets, also $G \cap (\phi^{-1}[A] \cup Q_0) \leq_w A$.

Let ρ be a complete metric on G. For each $s \in 2^{<\omega}$ and $\varepsilon > 0$ we define $s(\varepsilon) \in 2^{<\omega}$ as follows. Since $Q_0 \subseteq G$, $s^\frown\mathbf{0} \in G$, so there exists $t \in 2^{<\omega}$ of odd length such that for each $i < |t|$, $t(i) = 0$, and $\{x \in G : s^\frown t < x\} \subseteq B_\rho(s^\frown\mathbf{0}, \frac{1}{2}\varepsilon)$. Put $s(\varepsilon) = s^\frown t$, and note that for all $x, y \in G$, if $s(\varepsilon) < x, y$, then $\rho(x, y) < \varepsilon$. Inductively, define $\sigma\colon 2^{<\omega} \to 2^{<\omega}$ satisfying

(i) $\sigma(\langle\ \rangle) = \langle\ \rangle$;
(ii) $\sigma(s^\frown i) = \sigma(s)^\frown i$ if $f(s) = i$ or $i = 1$;
(iii) $\sigma(s^\frown 0) = (\sigma(s))(\frac{1}{|s|+1})$ if $f(s) = 1$ or $s = \langle\ \rangle$.

Intuitively, the induced mapping f_σ replaces the first element of each block of zeros of an element of 2^ω by a string of zeros of odd length, leaving the parity of the block unchanged. Thus, $f_\sigma[Q_0] \subseteq Q_0$, $f_\sigma[Q_1] \subseteq Q_1$, $f_\sigma[P] \subseteq P$ and $\phi f_\sigma(x) = \phi(x)$ for each $x \in P$.

Claim: If $x \in P$, then $f_\sigma(x) \in G$.

Indeed, for each n let $s_n \in 2^{<\omega}$ be such that $f(s_n) = 1$, $|s_n| \geq n$, $s_n < s_{n+1}$, and $s_n{}^\frown 0 < x$ (so s_n is an initial segment immediately preceding a block of zeros in x). Put $y_n = \sigma(s_n{}^\frown 0)^\frown\mathbf{0}$; then each $y_n \in Q_0 \subseteq G$. Also, if $m \geq n$, then $\sigma(s_n{}^\frown 0) = (\sigma(s_n))(\frac{1}{|s_n|+1}) < y_n, y_m$, so $\rho(y_n, y_m) < \frac{1}{|s_n|+1} \leq \frac{1}{n+1}$. Thus, $(y_n)_n$ is a ρ-Cauchy sequence, so it converges to some $y \in G$ by completeness of ρ. Clearly $\sigma(s_n) < y$ for each n, so $y = f_\sigma(x) \in G$.

It now easily follows that $f_\sigma^{-1}[G \cap (\phi^{-1}[A] \cup Q_0)] = \phi^{-1}[A] \cup Q_0$. Indeed, for "$\supseteq$", if $x \in Q_0$, then $f_\sigma(x) \in Q_0 \subseteq G$; and if $x \in \phi^{-1}[A]$, then $x \in P$, so $f_\sigma(x) \in G$ by the claim, and $\phi f_\sigma(x) = \phi(x) \in A$, so $f_\sigma(x) \in \phi^{-1}[A]$. For "$\subseteq$", if $f_\sigma(x) \in G \cap (\phi^{-1}[A] \cup Q_0)$, then either $f_\sigma(x) \in Q_0$ whence $x \in Q_0$, or $f_\sigma(x) \in \phi^{-1}[A] \subseteq P$ whence $\phi f_\sigma(x) = \phi(x) \in A$, so $x \in \phi^{-1}[A]$. Thus, $A \leq_w \phi^{-1}[A] \cup Q_0 \leq_w G \cap (\phi^{-1}[A] \cup Q_0)$. □

We now state our main lemma in the following form.

Lemma 3.3. *Let $\Gamma \subseteq \mathcal{P}(2^\omega)$ be a reasonably closed class of Borel sets, and suppose A is everywhere properly Γ and either first category or Baire. Then A contains a countable dense subset D such that for every relative $\mathbf{\Pi}^0_2$-set B of A, if $B \supseteq D$, then $B \approx A$.*

Proof. Let G be a $\mathbf{\Pi}^0_2$-set in 2^ω such that $G \supseteq Q_0$, and let U be a non-empty clopen subset of 2^ω. Since Γ is reasonably closed, $U \cap G \cap (\phi^{-1}[A] \cup Q_0) \leq_w A$. By the previous lemma, let $g\colon 2^\omega \to 2^\omega$ witness $A \leq_w G \cap (\phi^{-1}[A] \cup Q_0)$. Then $g^{-1}[U \cap G \cap (\phi^{-1}[A] \cup Q_0)] = g^{-1}[U] \cap A \equiv_w A$ since A is everywhere properly Γ, so $A \leq_w U \cap G \cap (\phi^{-1}[A] \cup Q_0)$. Thus, $G \cap (\phi^{-1}[A] \cup Q_0)$ is everywhere properly Γ. Suppose A is first category. Since ϕ is open, $\phi^{-1}[A]$ is first category as well, and this easily implies that $G \cap (\phi^{-1}[A] \cup Q_0)$ is first category. Suppose A is Baire. Since A has the Baire property in 2^ω, A contains a dense subset H which is $\mathbf{\Pi}^0_2$ in 2^ω. Again using the fact that ϕ is open, with domain P a $\mathbf{\Pi}^0_2$-set, it follows that $\phi^{-1}[H] \cap G$ is a dense absolute $\mathbf{\Pi}^0_2$-set in $G \cap (\phi^{-1}[A] \cup Q_0)$, so $G \cap (\phi^{-1}[A] \cup Q_0)$ is Baire. We conclude that A and $G \cap (\phi^{-1}[A] \cup Q_0)$ are everywhere properly Γ, and either both first category or both Baire, so by Theorem 2.2, $G \cap (\phi^{-1}[A] \cup Q_0) \approx A$. Applying this to $G = 2^\omega$ we obtain $h\colon \phi^{-1}[A] \cup Q_0 \approx A$ and clearly now $D = h[Q_0]$ is as required. □

4. The examples

Define subsets Y_0, Y_1 of $2^\omega \times (2^\omega)^\omega \approx 2^\omega$ by

$$Y_0 = Q_0 \times Q_0^\omega, Y_1 = Q_1 \times ((2^\omega)^\omega - Q_0^\omega).$$

Put

$$Y = Y_0 \cup Y_1$$

and

$$X = (\omega \times Y) \cup (\{\omega\} \times (2^\omega - Y))$$

in $(\omega + 1) \times 2^\omega \approx 2^\omega$. Towards an application of Theorem 2.2, define

$$\Gamma = \{A : \exists C_0, C_1 \in \mathbf{\Sigma}^0_2 : \exists A_0 \in \mathbf{\Pi}^0_3 : \exists A_1 \in \mathbf{\Sigma}^0_3 : \\ A = (A_0 \cap C_0) \cup (A_1 \cap C_1), C_0 \cap C_1 = \emptyset\}.$$

Lemma 4.1. *Both Γ and $\check{\Gamma}$ are reasonably closed.*

Proof. It is easily seen that if Γ is reasonably closed, then so is $\check{\Gamma}$. Since $\mathbf{\Sigma}^0_2$, $\mathbf{\Sigma}^0_3$, and $\mathbf{\Pi}^0_3$ are continuously closed, so is Γ. Let $A \in \Gamma$, say $A = (A_0 \cap C_0) \cup (A_1 \cap C_1)$ with C_0, C_1 disjoint $\mathbf{\Sigma}^0_2$-sets, $A_0 \in \mathbf{\Pi}^0_3$, and $A_1 \in \mathbf{\Sigma}^0_3$. Let C_0', C_1' be disjoint $\mathbf{\Sigma}^0_2$-sets reducing $\phi^{-1}[C_0] \cup Q_0 \cup Q_1, \phi^{-1}[C_1] \cup Q_0 \cup Q_1$, and put $A_0' = \phi^{-1}[A_0] \cup Q_0 \in \mathbf{\Pi}^0_3$ and $A_1' = \phi^{-1}[A_1] \cup Q_0 \in \mathbf{\Sigma}^0_3$. Then $\phi^{-1}[A] \cup Q_0 = (A_0' \cap C_0') \cup (A_1' \cap C_1') \in \Gamma$. □

Lemma 4.2. (a) *Y is everywhere properly Γ and first category.*
(b) *$2^\omega - Y$ is everywhere properly $\check{\Gamma}$ and Baire.*

Proof. (a) It is obvious that Y is first category. Let U be a non-empty clopen subset of $2^\omega \times (2^\omega)^\omega$. Put $A_0 = U \cap Y_0 \in \mathbf{\Pi}^0_3$, $A_1 = U \cap Y_1 \in \mathbf{\Sigma}^0_3$, and $C_0 = U \cap (Q_0 \times (2^\omega)^\omega) \in \mathbf{\Sigma}^0_2$, $C_1 = U \cap (Q_1 \times (2^\omega)^\omega) \in \mathbf{\Sigma}^0_2$; then $C_0 \cap C_1 = \emptyset$ and $U \cap Y = (A_0 \cap C_0) \cup (A_1 \cap C_1) \in \Gamma$. Suppose $U \cap Y \in \check{\Gamma}$; then $U - Y \in \Gamma$ whence also $(V \times W) - Y \in \Gamma$ for some non-empty clopen $V \subseteq 2^\omega$, $W \subseteq (2^\omega)^\omega$, say $(V \times W) - Y = (B_0 \cap D_0) \cup (B_1 \cap D_1)$ where B_0, B_1 are disjoint $\mathbf{\Sigma}^0_2$-sets, $D_0 \in \mathbf{\Pi}^0_3$

and $D_1 \in \mathbf{\Sigma}^0_3$. Write $B_0 = \bigcup_j B^0_j, B_1 = \bigcup_j B^1_j$ with each B^i_j compact. We claim that each $B^i_j \cap (V \times W) - Y$ is nowhere dense. Take $i = 0$ (the proof for $i = 1$ is the same). Suppose $V' \times W'$ is clopen in $V \times W$ such that $\emptyset \neq (V' \times W') - Y \subseteq B^0_j$. Let $x \in V' \cap Q_0$; then $(\{x\} \times W') - Y = (\{x\} \times W') - (\{x\} \times Q^\omega_0) = \{x\} \times (W' \cap (2^\omega)^\omega - Q^\omega_0) \approx 2^\omega - \mathbb{Q}^\omega$. However, $(\{x\} \times W') - Y$ is closed in $B_0 \cap D_0 \in \mathbf{\Pi}^0_3$ but $2^\omega - \mathbb{Q}^\omega \notin \mathbf{\Pi}^0_3$, a contradiction. We conclude that both $(V \times W) - Y$ and Y are first category, a contradiction. Part (b) follows easily from (a). □

Theorem 4.3. (a) *Y is a strongly homogeneous zero-dimensional absolute Borel set which does not admit the structure of a topological group.*

(b) *X is a non-homogeneous zero-dimensional absolute Borel set.*

Proof. (a) Strong homogeneity of Y follows from the previous lemmas and Theorem 2.2. Suppose Y has a topological group structure. By Lemma 3.3, let D be a countable subset of Y such that for every relative $\mathbf{\Pi}^0_2$-set B in Y, if $B \supseteq D$, then $B \approx Y$. Put $F = \bigcup_{d \in D} Y_1 d^{-1}$. Then $F \in \mathbf{\Sigma}^0_3$, so since $Y \notin \mathbf{\Sigma}^0_3$, there exists $x \in Y - F$. Then $xD \cap Y_1 = \emptyset$, so $D \subseteq x^{-1} Y_0$. Now Y_0 and hence $x^{-1}Y_0$ is a relative $\mathbf{\Pi}^0_2$-set of Y, so by assumption $Y_0 \approx x^{-1} Y_0 \approx Y$, a contradiction since $Y_0 \in \mathbf{\Pi}^0_3$ but $Y \notin \mathbf{\Pi}^0_3$.

(b) Suppose $h\colon X \to X$ maps $x = (\omega, z)$ to (n, y) with $n \in \omega$. Then for some clopen neighborhood U of x, $V = h[(\{\omega\} \times (2^\omega - Y)) \cap U] \subseteq \{n\} \times Y$. By Lemma 4.2(b), $V \notin \Gamma$. On the other hand, V is closed in X and hence in $\{n\} \times Y \in \Gamma$, a contradiction. □

The next theorem completes the proof of the properties of our examples.

Theorem 4.4. *Both $Y \times Y$ and $X \times X$ admit a topological group structure.*

Proof. Define $D_2(\mathbf{\Sigma}^0_3) = \{A : \exists A_0, A_1 \in \mathbf{\Sigma}^0_3 : A = A_1 - A_0\}$. The class $D_2(\mathbf{\Sigma}^0_3)$ is one of the so-called small Borel classes or difference classes that stratify the ambiguous Borel class $\mathbf{\Delta}^0_4$, and it is well known that these classes are non-self-dual (see [13] or [14]). It is easy to show that $D_2(\mathbf{\Sigma}^0_3)$ is reasonably closed. Suppose A is a space that is everywhere properly $D_2(\mathbf{\Sigma}^0_3)$ and first category. If $A = A_0 - A_1$ with $A_0, A_1 \in \mathbf{\Sigma}^0_3$, then $A \times A = (A_0 \times A_0) - ((A_1 \times A_0) \cup (A_0 \times A_1)) \in D_2(\mathbf{\Sigma}^0_3)$, which implies that $A \times A$ is everywhere properly $D_2(\mathbf{\Sigma}^0_3)$ and first category as well. Thus, $A \approx A \times A$ by Theorem 2.2, so A admits the structure of a topological group (in fact, of an ideal on ω) by [10]. We conclude that, in order to prove the theorem, it suffices to show that both $Y \times Y$ and $X \times X$ are everywhere properly $D_2(\mathbf{\Sigma}^0_3)$, as they are clearly first category.

Since $Y \in \Gamma$, we can write $Y = (A_0 \cap C_0) \cup (A_1 \cap C_1)$ with $A_0 \in \mathbf{\Pi}^0_3$, $A_1 \in \mathbf{\Sigma}^0_3$, and C_0, C_1 disjoint $\mathbf{\Sigma}^0_2$-sets. Put $B_0 = (A_1 \cap C_1) \cup C_0, B_1 = C_0 - A_0$; then $B_0, B_1 \in \mathbf{\Sigma}^0_3$, so $Y = B_0 - B_1 \in D_2(\mathbf{\Sigma}^0_3)$. As above we obtain $Y \times Y \in D_2(\mathbf{\Sigma}^0_3)$. Let U be a non-empty clopen subset of 2^ω. Clearly, $U \cap (Y \times Y) \in D_2(\mathbf{\Sigma}^0_3)$, so suppose $U \cap (Y \times Y) \in \check{D}_2(\mathbf{\Sigma}^0_3)$. Since U contains a non-empty basic clopen $V \times W$, and $V \cap Y \approx W \cap Y \approx Y$ by strong homogeneity, this implies $Y \times Y \in \check{D}_2(\mathbf{\Sigma}^0_3)$. Since $D_2(\mathbf{\Sigma}^0_3)$ is non-self-dual, there exists $A \in D_2(\mathbf{\Sigma}^0_3) - \check{D}_2(\mathbf{\Sigma}^0_3)$, so for this A we cannot have $A \leq_w Y \times Y$. However, let $A \in D_2(\mathbf{\Sigma}^0_3)$ be arbitary, say $A = A_0 - A_1$ with $A_0, A_1 \in \mathbf{\Sigma}^0_3$. By [12] (or using the Wadge Lemma and the remarks following Theorem 2.2), $A_0 \leq_w 2^\omega - \mathbb{Q}^\omega$ and $2^\omega - A_1 \leq_w \mathbb{Q}^\omega$. Since $2^\omega - \mathbb{Q}^\omega \equiv_w Y_1 \leq_w Y$ and $\mathbb{Q}^\omega \equiv_w Y_0 \leq_w Y$ ($Y_0, Y_1 \leq_w Y$ since $Y_0, Y_1 \in \Gamma = [Y]$), there exist $f, g\colon 2^\omega \to 2^\omega$ witnessing $A_0, 2^\omega - A_1 \leq_w Y$. Then $(f,g)\colon 2^\omega \to 2^\omega \times 2^\omega$ satisfies $(f,g)^{-1}[Y \times Y] = f^{-1}[Y] \cap g^{-1}[Y] = A_0 \cap (2^\omega - A_1) = A$. Thus, each element of $D_2(\mathbf{\Sigma}^0_3)$ is Wadge-reducible to $Y \times Y$, we have a contradiction, and we conclude that $Y \times Y$ is everywhere properly $D_2(\mathbf{\Sigma}^0_3)$.

If we write $Y = (A_0 \cap C_0) \cup (A_1 \cap C_1)$ as above, then $2^\omega - Y = (2^\omega - (A_0 \cap C_0)) - (A_1 \cap C_1) \in D_2(\mathbf{\Sigma}^0_3)$. We then easily obtain that $X \in D_2(\mathbf{\Sigma}^0_3)$ whence as before $X \times X \in D_2(\mathbf{\Sigma}^0_3)$. Since every non-empty clopen subset of $X \times X$ contains a clopen subset homeomorphic to $Y \times Y$, it immediately follows that $X \times X$ is everywhere properly $D_2(\mathbf{\Sigma}^0_3)$. □

Remark. In the above examples, we can replace $\mathbf{\Sigma}^0_3$,$\mathbf{\Pi}^0_3$ by any $\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi$ with $\xi \geq 3$, thus obtaining spaces of arbitrarily high Borel complexity with the same properties.

5. Wadge-minimality

The papers [16], [1] and [2], which answer the question of Arhangel'skiĭ for positive dimensions, all construct compact metrizable examples that are not just non-homogeneous but in fact rigid. So it is a natural question to ask whether such an example also exists in dimension zero. The answer is no: in fact, in van Engelen, Miller and Steel [9] it was shown that rigid absolute Borel sets do not exist at all. Thus, we must give up either rigidity or descriptive structure. If we give up the latter, we arrive at van Mill's question from [16] (still open) whether there exists any rigid zero-dimensional space with a homogeneous square. If we give up rigidity, then it becomes natural to ask for an example of minimal descriptive complexity. In this section we will show that Y is minimal among homogeneous spaces with a non-group square, and that X is minimal among non-homogeneous spaces with a homogeneous square (whence, in particular, X is minimal among non-homogeneous spaces whose squares admit a topological group structure).

As mentioned in section 1, we will presuppose knowledge of and adopt the notation from [14] and [5]. Let us also mention that a non-complete Borel group is necessarily first category; see, e.g., [6].

Lemma 5.1. *If $\mathbf{\Sigma}^0_3 \cup \mathbf{\Pi}^0_3 \subseteq \Gamma_u$ and $u(0) \geq 2$, then $Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3) \subseteq \Gamma_u$.*

Proof. Case 1: $\Gamma_u = D_\eta(\mathbf{\Sigma}^0_\xi)$. Then $\Gamma_u \supseteq D_2(\mathbf{\Sigma}^0_3)$. But $D_2(\mathbf{\Sigma}^0_3) \supseteq Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)$ by the proof of Theorem 4.4.

Case 2: $\Gamma_u = Sep(D_\eta(\mathbf{\Sigma}^0_\xi), \Gamma_{u^*})$. Then $\Gamma_u \supseteq Sep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)$ since $u^*(0) > u(0) = \xi \geq 2$. But $Sep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3) \supseteq Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)$: indeed, if $X = (A_0 \cap C_0) \cup (A_1 \cap C_1)$ as in Case 1, then $X = (A_0 \cap C_0) \cup ((A_1 \cap C_1) - C_0) \in Sep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)$.

Case 3: $\Gamma_u = Bisep(D_\eta(\mathbf{\Sigma}^0_\xi), \Gamma_{u_0}, \Gamma_{u_1})$. Then $\Gamma_u \supseteq Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)$ since $u_0(0) > u(0) = \xi \geq 2$.

Case 4: $\Gamma_u = SU(\mathbf{\Sigma}^0_\xi, \bigcup_n \Gamma_{u_n})$. Since $\sup u_n(0) > u(0) = \xi \geq 2$, we have $\mathbf{\Sigma}^0_3 \subseteq \Gamma_{u_n}$. But $\Gamma_{u_n} \subseteq \check{\Gamma}_{u_{n+1}}$, so $SU(\mathbf{\Sigma}^0_\xi, \bigcup_n \Gamma_{u_n}) \supseteq SU(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3 \cup \mathbf{\Pi}^0_3) \supseteq Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)$.

Case 5: $\Gamma_u = SD_\eta(\langle \mathbf{\Sigma}^0_\xi, \Gamma_{u_0} \rangle, \Gamma_{u_1})$. Then $\Gamma_u \supseteq \Gamma_{u_0} = SU(\mathbf{\Sigma}^0_\xi, \bigcup_n \Gamma_{v_n})$, which contains $Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)$ by Case 4. □

Since the class Γ of section 4 is $Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)$, the following theorem establishes minimality of our space Y.

Theorem 5.2. *Let Z be a homogeneous absolute Borel set which does not admit the structure of a topological group while $Z \times Z$ does admit such a structure. Then $Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3) \subseteq [Z]$.*

Proof. Suppose $Z \in D_2(\mathbf{\Sigma}^0_2)$. Then Z is discrete or one of $2^\omega, \omega \times 2^\omega, \omega^\omega, \mathbb{Q}, \mathbb{Q} \times 2^\omega$, or $\mathbb{Q} \times \omega^\omega$, so Z is in fact a group. If $Z \in \mathbf{\Delta}^0_3 - D_2(\mathbf{\Sigma}^0_2)$, then $Z \times Z$ is first category and by [7], $[Z \times Z] = D_\alpha(\mathbf{\Sigma}^0_2)$ for some indecomposable $\alpha < \omega_1$. Let U be non-empty and clopen in Z. Since Z is in fact strongly homogeneous, $U \in D_\alpha(\mathbf{\Sigma}^0_2) - D_2(\mathbf{\Sigma}^0_2)$ is homogeneous and first category, so by [5] $[U] = D_\beta(\mathbf{\Sigma}^0_2)$ for some $\beta \leq \alpha$. If $\beta < \alpha$, then by Lemmas 4.2 and 4.3 of [10], $U \times U \in D_\gamma(\mathbf{\Sigma}^0_2)$ for some $\gamma < \alpha$, contradicting the fact that $U \times U \approx Z \times Z$ by strong homogeneity of $Z \times Z$. Thus $[U] = D_\alpha(\mathbf{\Sigma}^0_2)$, so Z is first category and everywhere properly $D_\alpha(\mathbf{\Sigma}^0_2)$, and by Theorem 2.2 we again obtain the contradiction $Z \approx Z \times Z$, as $D_\alpha(\mathbf{\Sigma}^0_2)$ is easily seen to be reasonably closed. A similar argument shows that $[Z]$ cannot be $\mathbf{\Sigma}^0_3$ or $\mathbf{\Pi}^0_3$. Since Z is homogeneous, by [5] $[Z] \in \{\Gamma_u, \check{\Gamma}_u\}$ for some description u with $u(0) \geq 2$. Now apply Lemma 5.1, noting that $[Z] \neq \mathit{Bisep}(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)^\vee$ since Z is first category. □

We now turn to the minimality of our example X. Recall that Arhangel'skiĭ's question was for an example of a non-homogeneous space with a homogeneous square. Thus, we would like to establish that X is Wadge-minimal in *that* class, and not just in the restricted class where the square is assumed to be a group. We will show that this is indeed the case, albeit that we will have to admit the dual of $[X]$ as well. Indeed, as in section 4 we can show that $2^\omega - X$ is non-homogeneous and that $(2^\omega - Y) \times (2^\omega - Y)$ is everywhere properly $D_2(\mathbf{\Sigma}^0_3)$ whence easily $(2^\omega - X) \times (2^\omega - X)$ is everywhere properly $D_2(\mathbf{\Sigma}^0_3)$. Since $(2^\omega - X) \times (2^\omega - X)$ is Baire, it is homogeneous, but being non-complete it cannot be a group. In fact, as we will see, no generator of $[2^\omega - X]$ is first category, so only X is Wadge-minimal if the square is assumed to be a group. First, we determine the Wadge class of X.

Lemma 5.3. $[X] = \mathit{Sep}(\mathbf{\Sigma}^0_1, \mathit{Bisep}(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3))^\vee$.

Proof. Note that $((\omega + 1) \times 2^\omega) - X = ((\omega \times (2^\omega - Y)) \cap (\omega \times 2^\omega)) \cup ((\{\omega\} \times Y) - (\omega \times 2^\omega)) \in \mathit{Sep}(\mathbf{\Sigma}^0_1, \mathit{Bisep}(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3))$. Now suppose $X \in \mathit{Sep}(\mathbf{\Sigma}^0_1, \mathit{Bisep}(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3))$, say $X = (A_0 \cap C) \cup (A_1 - C)$ with $C \in \mathbf{\Sigma}^0_1$, $A_0 \in \mathit{Bisep}(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)^\vee$, and $A_1 \in \mathit{Bisep}(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)$. Since $X \notin \mathit{Bisep}(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)$, we have $A_0 \cap C \neq \emptyset$. Then $A_0 \cap C$, being open in X, intersects some $\{n\} \times Y$, contradicting Y being everywhere properly $\mathit{Bisep}(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)$. □

The proof that X is Wadge-minimal among non-homogeneous spaces with a homogeneous square is rather complicated. The main problem is to show that there are no examples in $\mathbf{\Delta}^0_3$ (Lemma 5.6).

Lemma 5.4. *Let $Z \subseteq 2^\omega$ be Borel. If $U \equiv_w Z$ for each non-empty clopen U in Z and $Z \times Z$ is homogeneous, then Z is homogeneous.*

Proof. If $Z \in D_2(\mathbf{\Sigma}^0_2)$, then Z has only one point, or $Z \times Z$ is one of $2^\omega, \omega \times 2^\omega, \omega^\omega, \mathbb{Q}, \mathbb{Q} \times 2^\omega$, or $\mathbb{Q} \times \omega^\omega$. It then follows from the characterizations of these spaces that $Z \approx Z \times Z$ is homogeneous. If $Z \notin D_2(\mathbf{\Sigma}^0_2)$, let u be a description (see [14]) such that $Z \in \{\Gamma_u, \check{\Gamma}_u, \mathbf{\Delta}(\Gamma_u)\}$. If $u(0) = 1$, then by the definition of Γ_u, always $U <_w Z$ for some non-empty clopen U in Z, so we must have $u(0) \geq 2$. If $\mathbf{\Delta}(D_\omega(\mathbf{\Sigma}^0_2)) \subseteq \Gamma_u$, then $[Z]$ is reasonably closed by [5], Lemma 4.2.17, so Z is homogeneous by Theorem 2.2. If $\Gamma_u \subseteq \mathbf{\Delta}(D_\omega(\mathbf{\Sigma}^0_2))$, then in fact $\Gamma_u = D_n(\mathbf{\Sigma}^0_2)$ for some $2 \leq n < \omega$, and $[Z] \in \{\Gamma_u, \check{\Gamma}_u\}$. In dealing with this situation, we extensively use the terminology and results from [5], sections 3.4 and 4.6 (see also [4]).

Case 1: $n = 2(k+1)$ for some $k < \omega$, $[Z] = D_n(\mathbf{\Sigma}^0_2)$. Then Z is $\mathcal{P}_{4k+1}$, nowhere $\mathcal{P}^2_{4k+2}$, so $Z \in \mathcal{X}_{4k+1}$ whence Z is homogeneous.

Case 2: $n = 2(k+1)+1$ for some $k < \omega$, $[Z] = \check{D}_n(\mathbf{\Sigma}^0_2)$. Then Z is $\mathcal{P}_{4(k+1)}$, nowhere $\mathcal{P}^2_{4k+3}$, so $Z \in \mathcal{X}_{4(k+1)}$ whence Z is homogeneous.

Case 3: $n = 2(k+1)$ for some $k < \omega$, $[Z] = \check{D}_n(\mathbf{\Sigma}^0_2)$. Then Z is $\mathcal{P}^2_{4k+2}$, nowhere $\mathcal{P}_{4k+1}$. If Z is $\mathcal{P}^1_{4k+2}$, then $Z \in \mathcal{X}^1_{4k+2}$, so Z is homogeneous; while if Z is nowhere $\mathcal{P}^1_{4k+2}$, then $Z \in \mathcal{X}^2_{4k+2}$, so Z is homogeneous as well. The only remaining possibility is that some non-empty clopen U in Z is $\mathcal{P}^1_{4k+2}$ (whence $U \in \mathcal{X}^1_{4k+2}$) but Z itself is not $\mathcal{P}^1_{4k+2}$. By [5], Theorem 3.4.15(b), Z contains a closed subset $A \in \mathcal{X}^2_{4(k-1)+3}$, $A \approx \mathbb{Q} \times B$, $B \in \mathcal{X}^2_{4(k-1)+2}$.

Claim: If $T \in \mathcal{X}^1_2, S \in \mathcal{X}^2_2$, then $T^{m+1} \in \mathcal{X}^1_{4m+2}, S^{m+1} \in \mathcal{X}^2_{4m+2}$ for each $m < \omega$.

We first show how we can use the claim to prove the lemma.

From their characterizations, it easily follows that $T \times 2^\omega \approx S$. For convenience, put $S^0 = 2^\omega$. Applying the claim we find that $U \times A \approx T^{k+1} \times \mathbb{Q} \times S^k \approx T^{k+1} \times \mathbb{Q} \times 2^\omega \times S^k \approx \mathbb{Q} \times S^{2k+1} \in \mathcal{X}^2_{4.2k+3}$. Since $U \times A$ is closed in $Z \times Z$, again by [5], Theorem 3.4.15, we have that $Z \times Z$ is not $\mathcal{P}^1_{4(2k+1)+2}$. On the other hand, applying the claim once again and using the fact that $Z \times Z$ is homogeneous whence strongly homogeneous, $Z \times Z \approx U \times U \approx T^{k+1} \times T^{k+1} = T^{2k+2} \in \mathcal{X}^1_{4(2k+1)+2}$, a clear contradiction.

Case 4: $n = 2(k+1)+1$ for some $k < \omega$, $[Z] = D_n(\mathbf{\Sigma}^0_2)$. Then Z is $\mathcal{P}^2_{4k+3}$, nowhere $\mathcal{P}_{4k+1}$. If Z is $\mathcal{P}^1_{4k+3}$, then $Z \in \mathcal{X}^1_{4k+3}$, so Z is homogeneous; while if Z is nowhere $\mathcal{P}^1_{4k+3}$, then $Z \in \mathcal{X}^2_{4k+3}$ is homogeneous. Assume U is a non-empty clopen subset of Z which is $\mathcal{P}^1_{4k+3}$ (whence $U \in \mathcal{X}^1_{4k+3}$), while Z is not $\mathcal{P}^1_{4k+3}$. By [5], Theorem 3.4.15(d), Z contains a closed subset $A \in \mathcal{X}^2_{4k+2}$. By the claim above, $U \times A \approx \mathbb{Q} \times T^{k+1} \times S^{k+1} \approx \mathbb{Q} \times S^{2k+2} \in \mathcal{X}^2_{4(2k+1)+3}$ and $U \times U \approx Z \times Z \approx \mathbb{Q} \times T^{k+1} \times \mathbb{Q} \times T^{k+1} \approx \mathbb{Q} \times T^{2k+2} \in \mathcal{X}^1_{4(2k+1)+3}$. Since $U \times A$ is closed in $Z \times Z$, $U \times A$ is $\mathcal{P}^1_{4(2k+1)+3}$, another contradiction.

It remains to establish the claim. We first prove by induction that if A_0 is $\mathcal{P}^i_{4(m-1)+2}$ and A_1 is $\mathcal{P}^i_2$, then $A_0 \times A_1$ is $\mathcal{P}^i_{4m+2}$. Here, "A is $\mathcal{P}^1_{-2}$" means "A has cardinality 1", and "A is $\mathcal{P}^2_{-2}$" means "A is compact". Clearly then, the statement holds for $m = 0$, so assume it holds for m, and A_0 is $\mathcal{P}^i_{4m+2}$. Then A_0 is $\mathcal{P}^i_{4(m-1)+3} \cup$ complete, so we can write $A_0 = \bigcup_{i<\omega} B_i \cup G$ where each B_i is $\mathcal{P}^i_{4(m-1)+2}$ and closed in $\bigcup_{i<\omega} B_i$, and G is complete. Note that $\bar{B}_i = B_i$ if $m = 0$, and $\bar{B}_i = B_i \cup (\bar{B}_i \cap G)$ is $\mathcal{P}^i_{4(m-1)+2} \cup$complete is $\mathcal{P}^i_{4(m-1)+2}$ if $m > 0$, so in fact we can assume that B_i is closed in A_0. Write $A_1 = \bigcup_{i<\omega} C_i \cup H$ where each C_i is $\mathcal{P}^i_{-2}$ and H is complete. Now $A_0 \times A_1 = \bigcup_{i<\omega}(B_i \times A_1) \cup \bigcup_{i<\omega}(A_0 \times C_i) \cup (G \times H)$, where $G \times H$ is complete and all $B_i \times A_1, A_0 \times C_i$ are closed in $A_0 \times A_1$. By the inductive hypothesis each $B_i \times A_1$ is $\mathcal{P}^i_{4m+2}$, and clearly so is each $A_0 \times C_i$. Thus, A_0 is $\mathcal{P}^i_{4m+3} \cup$ complete is $\mathcal{P}^i_{4(m+1)+2}$.

We now prove the claim by induction. The case $m = 0$ is trivial, so assume the claim holds for $m - 1$. By the above, T^{m+1} is $\mathcal{P}^1_{4m+2}$ and S^{m+1} is $\mathcal{P}^2_{4m+2}$. Since T is not complete, it contains a closed copy of $\mathbb{Q}$, so T^{m+1} contains a closed copy of $\mathbb{Q} \times T^m \in \mathcal{X}^1_{4(m-1)+3}$, thus T^{m+1} is not $\mathcal{P}_{4m}$ by [5], Theorem 3.4.15(a), hence nowhere $\mathcal{P}_{4m}$ by (strong) homogeneity. Since T^{m+1} is Baire, it easily follows that it is nowhere $\mathcal{P}_{4m+1}$, so $T^{m+1} \in \mathcal{X}^1_{4m+2}$. Similarly, S^{m+1} contains a closed

$\mathbb{Q} \times S^m \in \mathcal{X}^2_{4(m-1)+3}$, hence S^{m+1} is nowhere $\mathcal{P}^1_{4m+2}$. Since S^{m+1} is Baire, it is also nowhere $\mathcal{P}^1_{4m+3}$, so $S^{m+1} \in \mathcal{X}^2_{4m+2}$, and we are done. □

If $\Gamma_0 = [A_0], \Gamma_1 = [A_1]$ are Wadge classes, we denote by $\Gamma_0 \times \Gamma_1$ the Wadge class $[A_0 \times A_1]$. The following result is proved in [11].

Lemma 5.5. *Let $1 \leq \alpha, \xi < \omega_1$. Then $D_\alpha(\mathbf{\Sigma}^0_\xi) \times D_\alpha(\mathbf{\Sigma}^0_\xi) \subseteq (\check{D}_\alpha(\mathbf{\Sigma}^0_\xi) \times \check{D}_\alpha(\mathbf{\Sigma}^0_\xi))^\vee \subseteq D_\alpha(\mathbf{\Sigma}^0_\xi) \times \check{D}_\alpha(\mathbf{\Sigma}^0_\xi)$.*

The previous two lemmas allow us to exclude elements of $\mathbf{\Delta}^0_3$ from the possible examples.

Lemma 5.6. *Let Z be a non-homogeneous space such that $Z \times Z$ is homogeneous. Then $Z \notin \mathbf{\Delta}^0_3$.*

Proof. Clearly, we may assume that Z is a Wadge-minimal such space. Suppose that $Z \in \mathbf{\Delta}^0_3$. First note that $Z \notin D_2(\mathbf{\Sigma}^0_2)$, otherwise as in the proof of Lemma 5.4, $Z \approx Z \times Z$ is homogeneous. In particular, $Z \times Z$ is in fact strongly homogeneous. By Lemma 5.4, Z contains a non-empty clopen U with $U <_w Z$; then $U \times U$ is homogeneous, so U is homogeneous by minimality of Z. Thus, $U \in \mathbf{\Delta}^0_3 - D_2(\mathbf{\Sigma}^0_2)$, so by [5], $[U] \in \{D_\alpha(\mathbf{\Sigma}^0_2), \check{D}_\alpha(\mathbf{\Sigma}^0_2)\}$ for some $2 \leq \alpha < \omega_1$. Now Z is either first category or Baire. If Z is first category, then $[U] = D_\alpha(\mathbf{\Sigma}^0_2)$; then also $\check{D}_\alpha(\mathbf{\Sigma}^0_2) \subseteq [Z]$, so $\check{D}_\alpha(\mathbf{\Sigma}^0_2) \times \check{D}_\alpha(\mathbf{\Sigma}^0_2) \subseteq [Z \times Z] = [U \times U] = D_\alpha(\mathbf{\Sigma}^0_2) \times D_\alpha(\mathbf{\Sigma}^0_2) \subseteq (\check{D}_\alpha(\mathbf{\Sigma}^0_2) \times \check{D}_\alpha(\mathbf{\Sigma}^0_2))^\vee$, the final inclusion by Lemma 5.5. However, $\check{D}_\alpha(\mathbf{\Sigma}^0_2) \times \check{D}_\alpha(\mathbf{\Sigma}^0_2)$ is non-self-dual (for it is generated by a homogeneous space), and we have a contradiction. If Z is Baire, then similarly we obtain $[U] = \check{D}_\alpha(\mathbf{\Sigma}^0_2)$, and $D_\alpha(\mathbf{\Sigma}^0_2) \times \check{D}_\alpha(\mathbf{\Sigma}^0_2) \subseteq [Z \times Z] = [U \times U] = \check{D}_\alpha(\mathbf{\Sigma}^0_2) \times \check{D}_\alpha(\mathbf{\Sigma}^0_2) \subseteq (D_\alpha(\mathbf{\Sigma}^0_2) \times \check{D}_\alpha(\mathbf{\Sigma}^0_2))^\vee$ using Lemma 5.5 once more, and we have another contradiction. □

We can now prove minimality of X.

Theorem 5.7. *Let Z be a non-homogeneous absolute Borel set.*

(a) *If $Z \times Z$ is homogeneous, then $Sep(\mathbf{\Sigma}^0_1, Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)) \subseteq [Z] \cup [\check{Z}]$.*
(b) *If $Z \times Z$ is a topological group, then $Sep(\mathbf{\Sigma}^0_1, Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3))^\vee \subseteq [Z]$.*

Proof. (a) By Lemma 5.6, $Z \notin \mathbf{\Delta}^0_3$; and by Lemma 5.4, $U <_w Z$ for some non-empty clopen U in Z. Since $U \times U \approx Z \times Z$, it follows that $U \notin \mathbf{\Delta}^0_3$, and in fact it is easily seen that $U \notin \mathbf{\Sigma}^0_3 \cup \mathbf{\Pi}^0_3$. As above, we may assume that Z is minimal whence U is homogeneous. It then follows from [5] that $[U] \in \{\Gamma_u, \check{\Gamma}_u\}$ for some description u with $u(0) \geq 2$, so $Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3) \subseteq \Gamma_u$ by Lemma 5.1. First assume that Z is first category. We will show that $Z \notin Sep(\mathbf{\Sigma}^0_1, Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3))$, which implies that $Sep(\mathbf{\Sigma}^0_1, Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)) \subseteq [\check{Z}]$ by the Wadge Lemma. Suppose $Z \in Sep(\mathbf{\Sigma}^0_1, Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3))$, say $Z = (A_0 \cap C) \cup (A_1 - C)$, with $C \in \mathbf{\Sigma}^0_1$, $A_0 \in Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)^\vee$ and $A_1 \in Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)$. If $A_0 \cap C \in Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)$, then $Z \in Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)$, contradicting $U <_w Z$, so in fact $[A_0 \cap C] = Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)^\vee$. Being open in Z and non-empty, $(A_0 \cap C) \times (A_0 \cap C)$ is homogeneous, so since $A_0 \cap C <_w Z$, $A_0 \cap C$ is homogeneous by minimality of Z. This is easily seen to imply that $A_0 \cap C$ is Baire, contradicting first categoricity of Z. If Z is Baire, a similar argument shows that $Z \notin Sep(\mathbf{\Sigma}^0_1, Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3))^\vee$ whence $Sep(\mathbf{\Sigma}^0_1, Bisep(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_3)) \subseteq [Z]$.

(b) By (a), Z is not complete whence first category. Now use the proof of (a). □

6. Concluding remarks

The results of this note suggest two questions. First, does there exist a homogeneous space whose square is (necessarily homogeneous but) not a group? The answer is yes, rather trivially: since a non-complete group is first category, any homogeneous non-complete Baire space will do. A Wadge-minimal example is the space $T = X_2^1$ from [8], [5]; its Wadge class is $\check{D}_2(\mathbf{\Sigma}_2^0)$. A minimal first category example (not quite as trivially; see [6] or [7]) is $\mathbb{Q} \times T = X_3^1$, of Wadge class $D_3(\mathbf{\Sigma}_2^0)$. Second, does there exist a non-homogeneous space whose square is homogeneous but not a group? It follows from Theorem 5.7 that $2^\omega - X$ provides a Wadge-minimal positive answer to this question, since again $(2^\omega - X) \times (2^\omega - X)$ is not a group due to its not being first category. We state the first category counterpart as an open question.

Question. Does there exist a zero-dimensional, separable, metrizable (Borel) non-homogeneous first category space whose square is homogeneous but does not admit the structure of a topological group?

References

[1] F. D. Ancel and S. Singh, *Rigid finite dimensional fcompacta whose squares are manifolds*, Proc. Am. Math. Soc. 87 (1983), 342–346. MR **83m:**54074

[2] F. D. Ancel, P. F. Duvall and S. Singh, *Rigid 3-dimensional compacta whose squares are manifolds*, Proc. Am. Math. Soc. 88 (1983), 330–332. MR **84m:**54039

[3] A. V. Arhangel'skiĭ, *Structure and classification of topological spaces and cardinal invariants*, Russian Matm. Surveys 33 (1978), 33–96. MR **80i:**54005

[4] F. van Engelen, *Homogeneous Borel sets of ambiguous class two*, Trans. Am. Math. Soc. 290 (1985), 1–39. MR **87g:**54083

[5] F. van Engelen, *Homogeneous zero-dimensional absolute Borel sets*, CWI Tract 27, 1986.

[6] F. van Engelen, *On Borel groups*, Top. Appl. 35 (1990), 197–207. MR **91m:**54047

[7] F. van Engelen, *Zero-dimensional Borel groups of ambiguous class two*, Erasmus University Report 9017/B (1990).

[8] F. van Engelen and J. van Mill, *Borel sets in compact spaces: some Hurewicz-type theorems*, Fund. Math. 124 (1984), 271–286. MR **86j:**54067

[9] F. van Engelen, A. W. Miller and J. Steel, *Rigid Borel sets and better quasiorder theory*, Contemp. Math. 65 (1987), 199–222. MR **88e:**03079

[10] F. van Engelen, *On Borel ideals*, Ann. Pure Appl. Logic 70 (1994), 177-203.

[11] F. van Engelen, *Boolean operations on Borel Wadge classes*, in preparation.

[12] R. Engelking, W. Holsztyński and R. Sikorski, *Some examples of Borel sets*, Coll. Math. 15 (1966), 271–274.

[13] M. Lavrentieff, *Sur les sous-classes de la classification de M. Baire*, C. R. Acad. Sc. Paris 180 (1925), 111–114.

[14] A. Louveau, *Some results in the Wadge-hierarchy of Borel sets*, Cabal Seminar 79–81, Lect. Notes in Math. 1019 (1983), 28–55.

[15] A. Louveau and J. Saint-Raymond, *Borel classes and closed games: Wadge-type and Hurewicz-type results*, Trans. Am. Math. Soc. 304 (1987), 431–467. MR **89g:**03068

[16] J. van Mill, *A rigid space X for which $X \times X$ is homogeneous; an application of infinite-dimensional topology*, Proc. Am. Math. Soc. 83 (1981), 597–600. MR **82h:**54067

[17] J. R. Steel, *Analytic sets and Borel isomorphisms*, Fund. Math. 108 (1980), 83–88. MR **82b:**03091

[18] H. X. Zhou, *Homogeneity properties and power spaces,*, Ph. D. thesis, Wesleyan University, Middletown 1993.

Erasmus Universiteit, Econometrisch Instituut, Postbus 1738, 3000 DR Rotterdam, The Netherlands

E-mail address: engelen@wis.few.eur.nl