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A SUBMARTINGALE INEQUALITY

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Abstract. This paper extends Burkholder's inequality between a nonnegative submartingale and a process strongly differentially subordinate to it.

1. Statement of the inequality

Let $0 \le \alpha \le 1$ and $1 < p < \infty$. Suppose that $f = (f_n)_{n\ge 0}$ and $g = (g_n)_{n\ge 0}$ are adapted to a filtration $(\mathcal{F}_n)_{n\ge 0}$ of a probability space $(\Omega, \mathcal{F}, P)$. Here f is a nonnegative submartingale and g is $\mathbb{R}^\nu$-valued, where ν is a positive integer. With $f_n = d_0 + \cdots + d_n$ and $g_n = e_0 + \cdots + e_n$ $(n \ge 0)$ we assume that

$$|e_n| \le |d_n| \qquad (n \ge 0), \tag{1.1}$$

$$|\mathbb{E}(e_n|\mathcal{F}_{n-1})| \le \alpha \mathbb{E}(d_n|\mathcal{F}_{n-1}) \qquad (n \ge 1). \tag{1.2}$$

Then, with $\|f\|_p = \sup_{n\ge 0} \|f_n\|_p$, we have the inequality

$$\|g\|_p \le (r-1)\|f\|_p \tag{1.3}$$

where $r = \max\{(\alpha+1)p, p/(p-1)\}$ and r is best possible.

Remarks. 1. The case $\alpha = 1$ is shown in Burkholder's paper [3].

2. For martingales f and g that satisfy (1.1) and have values in a Hilbert space, the inequality (1.3) is shown in Burkholder's paper [2]. There $r = \max\{p, p/(p-1)\}$.

3. *Best possible* means that if $r-1$ in (1.3) is replaced by a strictly smaller number β, then the opposite inequality $\|g\|_p > \beta\|f\|_p$ holds for some probability space and some f and g as above.

2. Outline of the proof of the inequality

We may assume $\|f\|_p < \infty$. Also, we may assume

$$f_{n-1} > 0 \quad \text{and} \quad |g_{n-1} + te_n| > 0 \quad \text{for all } t \in \mathbb{R} \text{ and } n \ge 1. \tag{2.1}$$

Indeed, for each $0 < \varepsilon < 1$, the processes f^ε and g^ε, where $f^\varepsilon_n = f_n + \varepsilon$ and $g^\varepsilon = (g_n, \varepsilon)$, satisfy (2.1) and all the assumptions in Section 1. Here g^ε is a process in $\mathbb{R}^{\nu+1}$. Since $\|g\|_p \le \|g^\varepsilon\|_p$ and $\|f^\varepsilon\|_p \le \|f\|_p + \varepsilon$, the inequality (1.3) follows from $\|g^\varepsilon\|_p \le (r-1)\|f^\varepsilon\|_p$ as ε tends to 0.

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Put $S = (0,\infty) \times \mathbb{R}^\nu$ and define functions U and V on S by

$$U(x,y) = (|y| - (r-1)x)(x+|y|)^{p-1}, \tag{2.2}$$

$$V(x,y) = |y|^p - ((r-1)x)^p. \tag{2.3}$$

Then the inequality (1.3) follows from

$$\mathbb{E}V(f_n, g_n) \le 0 \quad \text{for all } n \ge 0 \tag{2.4}$$

which is a consequence of the inequalities:

$$\mathbb{E}V(f_n, g_n) \le p(1-1/r)^{p-1}\mathbb{E}U(f_n, g_n) \quad \text{for all } n \ge 0, \tag{2.5}$$

$$\mathbb{E}U(f_n, g_n) \le \mathbb{E}U(f_{n-1}, g_{n-1}) \quad \text{for all } n \ge 1, \tag{2.6}$$

$$\mathbb{E}U(f_0, g_0) \le 0. \tag{2.7}$$

3. Proof of the inequality

It remains to prove the inequalities (2.5), (2.6) and (2.7).

Proof of (2.5). Notice that (2.5) is proved if $V(x,y) \le p(1-1/r)^{p-1}U(x,y)$ for all $(x,y) \in S$, and that the latter inequality holds if and only if it holds with x replaced by $x/(x+|y|)$ and y replaced by $y/(x+|y|)$. Thus, it is enough to prove the inequality for $x \in (0,1]$ and $|y| = 1-x$, or, equivalently, to prove $F \le 0$ on $[0,1]$ where

$$F(x) = (1-x)^p - [(r-1)x]^p - p(1-rx)(1-1/r)^{p-1}. \tag{3.1}$$

If $0 < x < 1$, then

$$F'(x) = -p[(1-x)^{p-1} + (r-1)^p x^{p-1}] + rp(1-1/r)^{p-1}, \tag{3.2}$$

$$F''(x) = p(p-1)[(1-x)^{p-2} - (r-1)^p x^{p-2}]. \tag{3.3}$$

Notice that $0 < 1/r < 1$ and $F(1/r) = F'(1/r) = 0$.

If $p = 2$, then $F''(x) = 2[1-(r-1)^2] \le 0$ on $(0,1)$ because $r \ge (\alpha+1)2 \ge 2$. Thus F has the maximum at $x = 1/r$ which proves $F \le 0$ on $[0,1]$ in this case.

Now assume $1 < p < 2$. The strict concavity of $\log x$ implies that

$$\log 1 > a\log a + (1-a)\log(a+1) \quad \text{or} \quad (a+1)^{a-1} > a^a \quad \text{for } 0 < a < 1, \tag{3.4}$$

so $a = p-1$ gives $(p-1)^{p-1} < p^{p-2}$. Hence $r^{p-1} \ge (p/(p-1))^{p-1} > p$. Thus $F(1) = -(r-1)^p - p(1-r)(1-1/r)^{p-1} = (p - r^{p-1})(r-1)^p/r^{p-1} < 0$. Let x^* be the zero of F''. Computation gives $x^*[1+(r-1)^{p/(p-2)}] = 1, 0 < 1/r < x^* < 1$, and $F''(x) < 0$ if $0 < x < x^*$. Thus the maximum of F on $[0, x^*]$ is $F(1/r) = 0$. Also, $F''(x) > 0$ if $x^* < x < 1$. Thus F is convex on $[x^*, 1]$, hence $F \le 0$ on $[x^*, 1]$; because $F(x^*) \le 0$ and $F(1) < 0$.

The case $p > 2$ is handled similarly. Here $F(0) < 0$ follows from the inequality

$$(a+1)^{a-1} < a^a \quad \text{for } a > 1 \tag{3.5}$$

which follows from (3.4) upon replacing a by $1/a$.

Proof of (2.6). The proof depends on the inequality: *If (x,y) and $(x+h, y+k)$ belong to $(0,\infty) \times \mathbb{R}^\nu$ with $|h| \ge |k|$, and $y + tk \ne 0$ for all $t \in \mathbb{R}$, then*

$$U(x+h, y+k) \le U(x,y) + \varphi(x,y)h + \psi(x,y)\cdot k \tag{3.6}$$

where $\varphi(x,y) = U_x(x,y)$ *and* $\psi(x,y) = U_y(x,y)$, *that is,*

$$\varphi(x,y) = [(p-r)(x+|y|) - (p-1)rx](x+|y|)^{p-2},$$
$$\psi(x,y) = [p(x+|y|) - (p-1)rx](x+|y|)^{p-2}y/|y|.$$

For a proof of (3.6) in the case $\alpha = 1$, see [3]; the case $0 \le \alpha < 1$ requires only the further observation that $r \ge p/(p-1)$ and $r \ge p$.

Now (2.1) and (3.6) give

$$U(f_n, g_n) \le U(f_{n-1}, g_{n-1}) + \varphi(f_{n-1}, g_{n-1})\, d_n + \psi(f_{n-1}, g_{n-1}) \cdot e_n \tag{3.7}$$

where all the random variables are integrable: for example the last one is integrable because of the assumption (1.1) and the estimate $|\psi(x,y)| \le pr(x+|y|)^{p-1}$. Conditioning on $\mathcal{F}_{n-1}$ and using the assumption (1.2), one sees from (3.7) that

$$\begin{aligned}&\mathbb{E}U(f_n, g_n) - \mathbb{E}U(f_{n-1}, g_{n-1})\\ &\quad\le \mathbb{E}([\varphi(f_{n-1}, g_{n-1}) + \alpha|\psi(f_{n-1}, g_{n-1})|]\, \mathbb{E}(d_n|\mathcal{F}_{n-1})).\end{aligned}$$

Here f is a submartingale, hence $\mathbb{E}(d_n|\mathcal{F}_{n-1}) \ge 0$. Thus the inequality (2.6) follows from the inequality $\varphi(x,y) + \alpha|\psi(x,y)| \le 0$ which, by homogeneity, is equivalent to the inequality $G(x) \le 0$ if $0 \le x \le 1$, where

$$G(x) = (p-r) - (p-1)rx + \alpha|p - (p-1)rx|.$$

Clearly G is convex, hence it suffices to show $G(0) \le 0$ and $G(1) \le 0$. Here $G(0) = (\alpha+1)p - r \le 0$, and noting that $p - (p-1)r \le 0$, one has

$$\begin{aligned}G(1) &= (p-r) - (p-1)r - \alpha[p - (p-1)r]\\ &= -r + (1-\alpha)[p - (p-1)r] \le 0.\end{aligned}$$

This proves (2.6).

Proof of (2.7). Since $r \ge 2$ we have $|y| - (r-1)x \le |y| - x \le 0$ if $|y| \le x$. Hence (2.7) follows from (2.2), the assumption (1.1) which gives $|g_0| \le |f_0|$ when $n = 0$, and the nonnegativity of f_0.

4. Sharpness of the inequality

Case 1. Suppose that $1 < p \le (\alpha+2)/(\alpha+1)$. Then $r = p/(p-1)$ and so $r-1$ is the best constant in (1.3) since it is already the best possible constant if f is a nonnegative martingale as can be seen in (5.90) and (5.91) of [1].

Case 2. Here $p > (\alpha+2)/(\alpha+1)$ so $r = (\alpha+1)p$. Choose a small $\eta > 0$ so that $\eta(p-1)(\alpha+1) < 2$. Define $(x_n)_{n\ge1}$ and $(\pi_n)_{n\ge1}$ by

$$rx_n = 2 + n(\alpha+1)\eta \quad \text{and} \quad \pi_1 = \frac{1}{2}, \pi_{n+1} = \frac{1}{2}\prod_{k=1}^{n}\left(1 - \frac{\eta}{x_k}\right).$$

Now we define a filtration $(\mathcal{F}_n)_{n\ge0}$ on the Lebesgue probability space $[0,1)$ as follows: $\mathcal{F}_0 = \{\varnothing, [0,1)\}$ and for $n \ge 1$, $\mathcal{F}_{2n-1} = \mathcal{F}_{2n}$ is generated by the partition of $[0,1)$ determined by $0 < \pi_n < \pi_{n-1} < \cdots < \pi_1 < 1$.

Using the same notation for the interval $[a,b)$ and its characteristic function on $[0,1)$ we put

$$\begin{aligned} &d_0 = e_0 = [0,1), \quad d_1 = -e_1 = -\left[0,\tfrac{1}{2}\right) + \left[\tfrac{1}{2},1\right), \\ &d_{2n} = \eta[0,\pi_n), \quad e_{2n} = \alpha\, d_{2n}, \\ &d_{2n+1} = -e_{2n+1} = -\eta[0,\pi_{n+1}) + (x_n - \eta)[\pi_{n+1},\pi_n). \end{aligned}$$

Let $f_n = d_0 + \cdots + d_n$ and $g_n = e_0 + \cdots + e_n$. Then one checks that f and g are adapted to $(\mathcal{F}_n)$, that f is a nonnegative submartingale, and that (1.1) and (1.2) are satisfied. Also,

$$\begin{aligned} f_{2n+1} &= \sum_{1\le k\le n} x_k[\pi_{k+1},\pi_k) + 2\left[\tfrac{1}{2},1\right), \\ g_{2n+1} &= r x_n[0,\pi_{n+1}) + (r-1)\sum_{k=1}^{n} x_k[\pi_{k+1},\pi_k). \end{aligned}$$

Thus we get

$$\|g_{2n+1}\|_p \ge (r-1)\|f_{2n+1}\|_p \left(\frac{A_n}{A_n + 2^p}\right)^{1/p}$$

with $A_n = a_1 + \cdots + a_n$ where $a_n = |x_n|^p(\pi_n - \pi_{n+1})$.

Notice that each pair of stopped processes, f^{2n+1} and $g^{2n+1} = (g_{k\wedge(2n+1)})_{k\ge 0}$, satisfies all the assumptions of Section 1. Thus, in order to establish sharpness in Case 2, it is enough to show that $\lim_{n\to\infty} A_n = \infty$. Put $b = (\alpha+1)\eta$. By Taylor's formula, as $n \to \infty$,

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \left(1 + \frac{b}{2+nb}\right)^{p-1}\left(1 - \frac{r\eta}{2+nb}\right) \\ &= \left(1 + \frac{(p-1)b}{2+bn} + O\left(\left(\frac{b}{2+nb}\right)^2\right)\right)\left(1 - \frac{r\eta}{2+nb}\right) \\ &= \left(1 + \frac{p-1}{n} + O\left(\frac{1}{n^2}\right)\right)\left(1 - \frac{p}{n} + O\left(\frac{1}{n^2}\right)\right) \\ &= 1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right). \end{aligned}$$

Now recall the Gauss test for the convergence of a series of positive numbers:

if $a_n > 0$ *and* $\dfrac{a_{n+1}}{a_n} = 1 - \dfrac{\lambda}{n} + O(\dfrac{1}{n^2})$, *then* $\sum a_n < \infty$ *if and only if* $\lambda > 1$.

In our case, $\lambda = 1$ so $\sum a_n = \lim A_n = \infty$.

This completes the proof that $r-1$ is the best constant in the inequality (1.3).

Remarks. 1. The inequality (1.3) holds even for $\alpha > 1$. The proof in Section 3 works provided that $\alpha \le [1 + \sqrt{1+(p-1)^2}]/(p-1)$. For $\alpha > 1$ sharpness of r could not be established.

2. If $0 < \|f\|_p < \infty$, then the inequality (1.3) is strict unless $\alpha = 0$ and $p = 2$: this follows as in [3] because $r > 2$ unless $\alpha = 0$ and $p = 2$.

3. The inequality (1.3), or, more precisely, the function U in (2.2), leads to analogous inequalities for Itô processes and smooth functions on Euclidean domains. These follow as in [3]. For example, the condition $|\Delta v| \le |\Delta u|$ in [3] is replaced

by $|\Delta v| \leq \alpha|\Delta u|$. The inequality $\|v\|_p \leq (p^{**} - 1)\|u\|_p$ in [3] becomes $\|v\|_p \leq (r-1)\|u\|_p$.

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