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# COMPACT FLAT MANIFOLDS WITH HOLONOMY GROUP $\mathbf{Z}_2 \oplus \mathbf{Z}_2$

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ABSTRACT. In this paper we construct a family of compact flat manifolds, for all dimensions $n \geq 3$, with holonomy group isomorphic to $\mathbf{Z}_2^2$ and first Betti number zero.

## INTRODUCTION

In 1957 Calabi (see [Ca], [Wo], p.126) showed that the compact flat manifolds with zero first Betti number are the building blocks for all compact flat manifolds.

Hantzsche and Wendt (1935) constructed the only existing 3-dimensional compact flat manifold with first Betti number zero; this manifold has holonomy group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. Cobb ([Co], 1975) constructed a family of manifolds with these properties, for all dimensions $n \geq 3$.

It has been suggested that Cobb's family might exhaust all compact flat manifolds with $\beta_1 = 0$ and holonomy $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ ([HS], p.184).

In this paper we will construct a family of compact flat manifolds $\mathcal{M}_n$ with $\beta_1 = 0$ and holonomy group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, which includes Cobb's family as a (relatively small) subclass. More precisely, we shall prove (see Theorem 3.5) that if $c_n$ denotes the cardinality of Cobb's family, and $b_n$ that of $\mathcal{M}_n$, then $c_n \sim C\, n^2$ and $b_n \sim B\, n^5$ as $n \to \infty$, with $C = \frac{1}{2^2 3}$ and $B = \frac{1}{2^7 3^2 5}$. The first manifold not in Cobb's family appears in dimension 5.

The main tool in our construction is a result of Dotti-Miatello and Miatello ([DM]) where an explicit general procedure for the construction of compact flat manifolds with holonomy $\mathbf{Z}_2^k$ is given.

We notice that the case of holonomy group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ is the simplest to be considered, since it is well known (see [HS]) that no flat manifold with a cyclic holonomy group can have zero first Betti number.

## 1. PRELIMINARIES

If $M$ is a compact flat Riemannian manifold of dimension $n$ then $M \simeq \Gamma \setminus \mathbf{R}^n$ where $\Gamma$, the fundamental group of $M$, is a torsion-free discrete cocompact subgroup of $\mathrm{I}(\mathbf{R}^n) = \mathrm{O}(n) \ltimes \mathbf{R}^n$. These groups are usually called Bieberbach groups.

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By Bieberbach's first theorem, for such a $\Gamma$, there exists an exact sequence

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

where $\Lambda$ is the only maximal free abelian normal subgroup of $\Gamma$ and $G$ is a finite subgroup of $\mathrm{O}(n)$. The group $G$ is called the *point group* of $\Gamma$ and is isomorphic to the holonomy group of the manifold $M$. Furthermore, Bieberbach also proved that, if $\Gamma$ and $\Gamma'$ are two isomorphic discrete cocompact subgroups of $\mathrm{I}(\mathbf{R}^n)$ then there exist $A \in \mathrm{Gl}(n, \mathbf{R})$ and $v \in \mathbf{R}^n$ such that $AL_v \Gamma L_{-v} A^{-1} = \Gamma'$. This implies that if $M$ and $M'$ are compact flat Riemannian manifolds with isomorphic fundamental groups, then $M$ is diffeomorphic to $M'$.

On the other hand, the exact sequence above induces an action of $G$ on $\Lambda$, which defines an $n$-dimensional integral representation of $G$. If $\Gamma$ and $\Gamma'$ are isomorphic, where the isomorphism is given by $AL_v$, then $A : \Lambda \to \Lambda'$, the point groups $G$ and $G'$ are isomorphic, and the holonomy representations $\rho$ and $\rho'$ are related by

$$\rho'_{AgA^{-1}} = A\, \rho_g\, A^{-1}$$

for any $g$ in $G$. In this case we will say that $\rho$ and $\rho'$ are $A$-equivalent.

We state a particular case of a proposition proved in [DM] which gives a procedure to construct Bieberbach groups, of arbitrary dimensions, with point group isomorphic to $\mathbf{Z}_2^2$. Moreover, this proposition characterizes all such groups.

**Proposition 1.1.** *Let $B_1$ and $B_2$ in $O(n)$ be such that $\langle B_1, B_2 \rangle \simeq \mathbf{Z}_2^2$. Let $\Lambda$ be a lattice in $\mathbf{R}^n$ stable by $B_1$ and $B_2$ and let $b_1$ and $b_2$ be chosen so that*

*(i)* $(B_1 - I)b_2 - (B_2 - I)b_1 \in \Lambda$;

*(ii)* $(B_i + I)b_i \in \Lambda - (B_i + I)\Lambda,\ i = 1, 2$;

*(iii)* $(B_iB_j + I)(B_ib_j + b_i) \in \Lambda - (B_iB_j + I)\Lambda,\ i \neq j$.

*Then $\Gamma = \langle B_1L_{b_1}, B_2L_{b_2}, \Lambda \rangle$ is a Bieberbach group with translation lattice $\Lambda$ and $M = \Gamma \setminus \mathbf{R}^n$ has holonomy group $\mathbf{Z}_2^2$.*

*Conversely, if $\Gamma$ is a Bieberbach group with point group isomorphic to $\mathbf{Z}_2^2$ and translation lattice $\Lambda$, then $\Gamma \simeq \langle B_1L_{b_1}, B_2L_{b_2}, \Lambda \rangle$ with $\langle B_1, B_2 \rangle \simeq \mathbf{Z}_2^2$ where, furthermore, $B_1, B_2, b_1$ and $b_2$ satisfy conditions (i)–(iii).*

We will make use of this proposition in sections 2 and 3.

## 2. Cobb's family

We begin this section with an application of Proposition 1.1, constructing a family of Bieberbach groups with point group $\mathbf{Z}_2^2$ and trivial center. This family will be isomorphic to the one constructed by Cobb in [Co] and will be used in section 3.

**Example 2.1.** Let $B_1$ and $B_2$ be the $m \times m$ diagonal matrices defined by

$$(B_1)_{ii} = \begin{cases} -1, & \text{if } 1 \leq i \leq m - m_1, \\ 1, & \text{if } m - m_1 < i \leq m, \end{cases} \qquad (B_2)_{ii} = \begin{cases} 1, & \text{if } 1 \leq i \leq m_2, \\ -1, & \text{if } m_2 < i \leq m, \end{cases}$$

where $m_1 + m_2 < m$. In addition, we choose $b_1 = \frac{e_m}{2},\ b_2 = \frac{e_1 + e_{m-m_1}}{2}$ in $\mathbf{R}^m$ and we let $\Lambda$ be the canonical lattice. It is not difficult to check that the conditions in Proposition 1.1 are satisfied. By varying $m_1$ and $m_2$ in such a way that $1 \leq m_1 \leq m_2$ and $m_1 + 2m_2 \leq m$, one obtains a family of Bieberbach groups, one for each pair $(m_1, m_2)$, which we shall denote by $\Gamma_{(m_1, m_2)}$.

Since $Z(\Gamma) = \Lambda^G$, where $G = \langle B_1, B_2 \rangle$, the condition $m_1 + m_2 < m$ ensures that the constructed Bieberbach groups have a trivial center. Recall that for $M = \Gamma \backslash \mathbf{R}^n$ we have $\beta_1(M) = \operatorname{rank} Z(\Gamma)$.

We now prove that a different choice of $b_1$ and $b_2$ does not affect the isomorphism class of $\Gamma_{(m_1,m_2)}$.

**Lemma 2.2.** *If $B_1$ and $B_2$ are as in Example 2.1, $\widetilde{b_1}$ and $\widetilde{b_2}$ are such that the conditions of Proposition 1.1 are satisfied and $\Gamma' = \langle B_1 L_{\widetilde{b_1}}, B_2 L_{\widetilde{b_2}}, \Lambda \rangle$, then $\Gamma' \simeq \Gamma_{(m_1,m_2)}$.*

*Proof.* We consider $\Gamma = \langle \alpha, \beta, \Lambda \rangle$ and $\widetilde{\Gamma} = \langle \widetilde{\alpha}, \widetilde{\beta}, \Lambda \rangle$, where $\alpha = B_1 L_{b_1}$, $\beta = B_2 L_{b_2}$, $\widetilde{\alpha} = B_1 L_{\widetilde{b_1}}$, $\widetilde{\beta} = B_2 L_{\widetilde{b_2}}$ and $\Lambda$ is the canonical lattice. It follows from Proposition 1.1 that the most general form for $\widetilde{b_1}$ and $\widetilde{b_2}$ is

$$\widetilde{b_1} = (x_1, \ldots, x_{m_2}, x_{m_2+1}, \ldots, x_{m-m_1}, \frac{1}{2}\delta_{m-m_1+1}, \ldots, \frac{1}{2}\delta_m)$$

$$\widetilde{b_2} = (\frac{1}{2}\delta_1, \ldots, \frac{1}{2}\delta_{m_2}, \frac{1}{2}\delta_{m_2+1} + x_{m_2+1}, \ldots, \frac{1}{2}\delta_{m-m_1} + x_{m-m_1}, x_{m-m_1+1}, \ldots, x_m)$$

with $\delta_i = 0$ or $1$ $\forall i$ and $\sum_{i=1}^{m_2} \delta_i \geq 1$, $\sum_{i=m_2+1}^{m-m_1} \delta_i \geq 1$, $\sum_{i=m-m_1+1}^{m} \delta_i \geq 1$. It is clear that we can choose $x_i \in [0,1)$ $\forall i$.

We must find $A \in \mathrm{Gl}(m, \mathbf{Z})$ and $v \in \mathbf{R}^m$ such that $A L_v \Gamma L_{-v} A^{-1} = \widetilde{\Gamma}$. If

$$A = \begin{pmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{pmatrix},$$

where $A_1$, $A_2$ and $A_3$ are square matrices of dimensions $m_2$, $m - m_1 - m_2$ and $m_1$ respectively, then clearly $A$ commutes with $B_1$ and $B_2$. Hence the result will follow if we can solve the following equations:

$$A(B_1 - I)v + Ab_1 \equiv \widetilde{b_1} \pmod{\Lambda},$$

$$A(B_2 - I)v + Ab_2 \equiv \widetilde{b_2} \pmod{\Lambda}.$$

We divide this system of equations into three parts, each one corresponding to the matrices $A_i$. We denote $v = (v_1, v_2, v_3)$, $b_1 = ((b_1)_1, (b_1)_2, (b_1)_3)$ and similarly for $\widetilde{b_1}$, $b_2$ and $\widetilde{b_2}$.

We choose $a_{i1} = \delta_i$ for $i = 1, \ldots, m_2$. Recall that there is at least one $\delta_i \neq 0$, so we may assume $\delta_j \neq 0$. Then we complete $A_1$ as follows:

$$A_1 = \left(\begin{array}{ccc|ccc} \delta_1 & & & 1 & & \\ \vdots & & & & \ddots & \\ \delta_{j-1} & & & & & 1 \\ \hline 1 & & & & & \\ \vdots & \ddots & & & & \\ \delta_{m_2} & & 1 & & & \end{array}\right),$$

where the blank spaces are zeros. Notice that $A_1 \in \mathrm{Gl}(m_2, \mathbf{Z})$. Now we calculate $v_1$ by $-2A_1v_1 = (\widetilde{b_1})_1$.

In an analogous way we obtain $A_3$ and $v_3$.

For the second set of equations we choose $a_{i,(m-m_1)} = \delta_i$, for $i = m_2+1, \dots, m-m_1$; assuming $\delta_l = 1$ we complete $A_2$ as follows:

$$A_2 = \left(\begin{array}{ccc|ccc} & & & 1 & & \delta_{m_2+1} \\ & & & & \ddots & \vdots \\ & & & & & 1 \\ \hline 1 & & & & & \delta_{l+1} \\ & \ddots & & & & \vdots \\ & & 1 & & & \delta_{m-m_1} \end{array}\right),$$

where the blank spaces are zeros and thus $A_2 \in \mathrm{Gl}(m - m_1 - m_2, \mathbf{Z})$. Then $v_2$ is the solution to $-2A_2v_2 = (\widetilde{b_1})_2$. □

We now show that the above family coincides, up to isomorphism, with the one given by Cobb. For this, let $X, Y, r_i, s_i, t_i, u_i$ be as in [Co], with $0 \leq i \leq m-1$. We have that $\mathcal{V} = \{u_i\}$ generates a lattice $\Lambda$ of dimension $m$. We note that the vectors $u_i$ are orthogonal to each other and all of them have the same length. One verifies that $X$ and $Y$ act on the basis $\mathcal{V}$ by

$$X(r_i) = r_i,\ X(s_i) = -s_i,\ X(t_i) = -t_i; \quad Y(r_i) = -r_i,\ Y(s_i) = s_i,\ Y(t_i) = -t_i.$$

After reordering, if necessary, we may assume that

$$\mathcal{V} = \{r_0, \dots, r_{m_2-1}, s_0, \dots, s_{m-m_2-m_1-1}, t_0, \dots, t_{m_1-1}\}$$

and thus $X = B_2$, $Y = B_1B_2$ and $XY = B_1$, where $B_1, B_2$ are as in Example 2.1. We also set $b_1 = \frac{s_0-t_0}{2}$ and $b_2 = \frac{r_0+t_0}{2}$. If we let $m_3 = m - m_1 - m_2$, the conditions on $(m_1, m_2)$ in Example 2.1 are equivalent to those used by Cobb, namely $1 \leq m_1 \leq m_2 \leq m_3$ and $m_1 + m_2 + m_3 = m$. By the lemma above, it follows that for each such choice of $(m_1, m_2)$, the group defined by Cobb is isomorphic to $\Gamma_{(m_1,m_2)}$. Furthermore, it is proved in [Co] that if $(m_1, m_2) \neq (m_1', m_2')$ then $\Gamma_{(m_1,m_2)} \not\simeq \Gamma_{(m_1',m_2')}$, by observing that the associated integral representations of the respective point groups are not $A$-equivalent for any $A \in \mathrm{Gl}(n, \mathbf{R})$.

We shall denote the family in Example 2.1 (i.e. Cobb's family) by

$$\mathcal{C}_m = \{\Gamma_{(m_1,m_2)} : 1 \leq m_1 \leq m_2,\ m_1 + 2m_2 \leq m\}.$$

*Remark 1.* Notice that by choosing $b_1 = \frac{e_2-e_3}{2}$ and $b_2 = \frac{e_1+e_3}{2}$, in Example 2.1, one obtains Cobb's family without making use of Lemma 2.2. Our election was made in order to simplify the construction in the following section.

*Remark 2.* As a consequence of Lemma 2.2 and the converse assertion in Proposition 1.1, it follows that any Bieberbach group with trivial center, canonical translation lattice $\Lambda$ and point group $G \simeq \mathbf{Z}_2^2$ acting diagonally on $\Lambda$, is isomorphic to $\Gamma_{(m_1,m_2)}$, for some $(m_1, m_2)$, i.e. it lies in Cobb's class.

## 3. A new family

If $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, then $\{\pm I, \pm J\}$ defines an integral representation of $\mathbf{Z}_2^2$. For $n \geq 3$, we will consider the $n \times n$ matrices

$$B_1 = \begin{pmatrix} D_1 & & & \\ & A_1^1 & & \\ & & \ddots & \\ & & & A_1^k \end{pmatrix}, \quad B_2 = \begin{pmatrix} D_2 & & & \\ & A_2^1 & & \\ & & \ddots & \\ & & & A_2^k \end{pmatrix},$$

where $D_1$ and $D_2$ are the diagonal matrices used in Example 2.1 and $A_i^j$ are $2 \times 2$ matrices, each one equal to one of $J$, $-J$ or $-I$, and such that the blocks $A_1^j$ and $A_2^j$ are different, for each $j = 1, \ldots, k$.

If $m$ is the dimension of $D_1$ and $D_2$, then $n = m + 2k$. Now, we take $b_1 = \frac{e_m}{2}$ and $b_2 = \frac{e_1 + e_{m-m_1}}{2}$ in $\mathbf{R}^n$. If $\alpha = B_1 L_{b_1}$, $\beta = B_2 L_{b_2}$ and $\Lambda$ is the canonical lattice, then it is easy to see, by using Proposition 1.1, that $\Gamma = \langle \alpha, \beta, \Lambda \rangle$ is a Bieberbach group with point group $\mathbf{Z}_2^2$. Moreover, as in Example 2.1, the corresponding manifold has first Betti number zero, because of the condition imposed on the $j$-th blocks of $B_1$ and $B_2$.

Let $\mathcal{B}_{n,k}$ be the collection of all Bieberbach groups constructed in this manner, with $n, k$ fixed, and let $\mathcal{B}_n$ be the union of all $\mathcal{B}_{n,k}$ with $0 \leq k \leq [(n-3)/2]$. Notice that $\mathcal{B}_{n,0} = \mathcal{C}_n$, Cobb's family.

We now determine $\mathrm{H}_1(M, \mathbf{Z}) \simeq \frac{\Gamma}{[\Gamma, \Gamma]}$, where $M = \Gamma \setminus \mathbf{R}^n$ and $\Gamma \in \mathcal{B}_{n,k}$.

Since $[\alpha, L_{e_i}] = L_{-2e_i}$ for $i = 1, 2, \ldots, m - m_1$ and $[\beta, L_{e_i}] = L_{-2e_i}$ for $i = m_2 + 1, \ldots, m$, then $L_{2e_i} \in [\Gamma, \Gamma]$, for $i = 1, 2, \ldots, m$.

We observe that for each $j = 1, \ldots, k$,

$$[\alpha, L_{e_{m+2j-1}}] = \begin{cases} L_{e_{m+2j} - e_{m+2j-1}}, & \text{if } A_1^j = J, \\ L_{-e_{m+2j} - e_{m+2j-1}}, & \text{if } A_1^j = -J, \\ L_{-2e_{m+2j-1}}, & \text{if } A_1^j = -I, \end{cases}$$

$$[\alpha, L_{e_{m+2j}}] = \begin{cases} L_{e_{m+2j-1} - e_{m+2j}}, & \text{if } A_1^j = J, \\ L_{-e_{m+2j-1} - e_{m+2j}}, & \text{if } A_1^j = -J, \\ L_{-2e_{m+2j}}, & \text{if } A_1^j = -I, \end{cases}$$

and similarly with $\beta$ instead of $\alpha$. Thus

$$\begin{aligned} \langle [\alpha, L_{e_{m+2j-1}}], [\beta, L_{e_{m+2j-1}}] \rangle &= \langle [\alpha, L_{e_{m+2j}}], [\beta, L_{e_{m+2j}}] \rangle \\ &= \langle L_{e_{m+2j} - e_{m+2j-1}}, L_{-e_{m+2j} - e_{m+2j-1}} \rangle. \end{aligned}$$

Furthermore, $\alpha^2 = L_{e_m}$, $\beta^2 = L_{e_1}$ and $[\alpha, \beta] = L_{-e_1 + e_{m-m_1} + e_m}$. Putting together all this information we obtain

$$\mathrm{H}_1(M, \mathbf{Z}) \simeq \mathbf{Z}_4^2 \oplus \mathbf{Z}_2^{m-3+k}.$$

As a consequence of this calculation and Bieberbach's theorem we have:

**Proposition 3.1.** *If $\Gamma_i$ are $n$-dimensional Bieberbach groups in $\mathcal{B}_{n,k_i}$, $i = 1, 2$, with $k_1 \neq k_2$, then $\Gamma_1$ and $\Gamma_2$ are not isomorphic.*

We now introduce another invariant to distinguish groups in $\mathcal{B}_{n,k}$, with fixed $n$ and $k$.

Let $B \in \mathrm{O}(n)$, preserving the canonical lattice $\Lambda$. Let $\Lambda^+$ be a sublattice of maximal rank $t^+$ of $\Lambda$, such that $B$ acts on $\Lambda^+$ as the identity and $\Lambda^+$ admits a $B$-invariant complement $\Delta^+$. Similarly we define $t^-$, $\Lambda^-$ and $\Delta^-$ for minus identity instead of identity.

**Lemma 3.2.** *If $B' = ABA^{-1}$ with $A \in \mathrm{Gl}(n, \mathbf{Z})$ and $t'^{\pm}$ are defined as $t^{\pm}$ with $B'$ in place of $B$, then $(t'^+, t'^-) = (t^+, t^-)$.*

*Proof.* Since $\Lambda = \Lambda^+ \oplus \Delta^+$ is a $B$-invariant decomposition, then $B = B|_{\Lambda^+} \oplus B|_{\Delta^+}$. So $B' \circ A|_{\Lambda^+} = A|_{\Lambda^+}$ and therefore $B'$ acts as the identity on $A(\Lambda^+)$. But $A(\Delta^+)$ is also $B'$-invariant, and $\Lambda' = \Lambda = A(\Lambda^+) \oplus A(\Delta^+)$ implies that $t^+ \leq t'^+$. By exchanging the roles of $t^+$ and $t'^+$ we get that $t'^+ \leq t^+$, so $t'^+ = t^+$. The same argument proves the assertion for $t^-$ and $t'^-$. □

For each $\Gamma \in \mathcal{B}_{n,k}$, $\Gamma = \langle B_1 L_{b_1}, B_2 L_{b_2}, \Lambda \rangle$, set $B_3 = B_1 B_2$ and for $1 \leq i \leq 3$ let $\Lambda_i^+$ be a sublattice of maximal rank $t_i^+$ of $\Lambda$, the canonical lattice, with $B_i$ acting on $\Lambda_i^+$ as the identity and such that $\Lambda_i^+$ admits a $B_i$-invariant complement $\Delta_i^+$. We define $\Lambda_i^-$ and $\Delta_i^-$ similarly.

The following triple of pairs will be important in the sequel. We shall denote

$$t_\Gamma = \big((t_1^+, t_1^-), (t_2^+, t_2^-), (t_3^+, t_3^-)\big).$$

**Corollary 3.3.** *If $\Gamma$, $\Gamma' \in \mathcal{B}_n$ and $\Gamma \simeq \Gamma'$, then $t_\Gamma = t_{\Gamma'}$, up to permutation.*

*Proof.* Recall that there exist $A \in \mathrm{Gl}(n, \mathbf{Z})$ and $v \in \mathbf{R}^n$ such that $AL_v \Gamma L_{-v} A^{-1} = \Gamma'$, hence $A\langle B_1, B_2\rangle A^{-1} = \langle B_1', B_2'\rangle$. Thus Lemma 3.2 clearly implies the corollary. □

Let $k_i = \big|\{j : A_i^j = -I\}\big|$ for $i = 1, 2$, and $k_3 = \big|\{j : A_1^j A_2^j = -I\}\big|$. Notice that $k_1 + k_2 + k_3 = k$.

Let $\Lambda_i^+ = \langle e_j : (B_i)_{jj} = 1,\ 1 \leq j \leq m\rangle$, for $i = 1, 2, 3$, and $\Lambda_i^- = \langle e_j : (B_i)_{jj} = -1,\ 1 \leq j \leq n\rangle$. It is not difficult to see that $\Lambda_i^+$ (resp. $\Lambda_i^-$) is, in fact, a sublattice of $\Lambda$ of maximal rank $m_i$ (resp. $m - m_i + 2k_i$), with $B_i$ acting as the identity (resp. minus identity) and admitting a $B_i$-invariant complement. Indeed, this fact can be proved by using ideas in [Re] by considering the groups $\frac{\mathrm{Ker}(B_i \pm I)}{\mathrm{Im}(B_i \mp I)}$ for $i = 1, 2$, which are invariant, up to isomorphism, under equivalence. Thus we have

$$t_\Gamma = \big((m_1, m - m_1 + 2k_1), (m_2, m - m_2 + 2k_2), (m_3, m - m_3 + 2k_3)\big).$$

The next lemma will be central in the proof of the main result.

**Lemma 3.4.** *Let $\Gamma, \Gamma' \in \mathcal{B}_n$. Then $\Gamma \simeq \Gamma'$ if and only if $t_\Gamma = t_{\Gamma'}$, up to permutation.*

*Proof.* The necessary condition is Corollary 3.3. Conversely, if $t_\Gamma = t_{\Gamma'}$, up to permutation, then there exists $\sigma \in \mathrm{S}_3$ such that $m_i = m'_{\sigma(i)}$ and $k_i = k'_{\sigma(i)}, 1 \leq i \leq 3$. By a permutation of the first $m$ elements of the basis of $\Lambda'$ (the canonical lattice), we can modify, if necessary, the diagonal parts, $D_i'$, of $B_i'$ so that $D_i = D'_{\sigma(i)}, 1 \leq i \leq 3$. In the same way, we can make a permutation of the last $2k$ elements of the basis of $\Lambda'$ so that $A_i^j = -I$ if and only if ${A'}_{\sigma(i)}^j = -I$, for $j = 1, \ldots, k$. Since this change of basis of $\Lambda'$ is made by conjugation by a unimodular matrix, we may assume that $D_i = D_i'$ and $A_i^j = -I$ if and only if ${A'}_i^j = -I$, for $1 \leq j \leq k, 1 \leq i \leq 3$.

The proof of Lemma 2.2 shows that there exist a matrix $A \in \mathrm{Gl}(m, \mathbf{Z})$ and a vector $v \in \mathbf{R}^m$ such that conjugation by $AL_v$ takes $D_i L_{b_i}$ into $D_i' L_{b_i'}$.

We now set

$$C = \begin{pmatrix} A & & & \\ & C_1 & & \\ & & \ddots & \\ & & & C_k \end{pmatrix},$$

where $C_j = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ if $A_1^j = -A'^j_1$ or $A_2^j = -A'^j_2$, and $C_j = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ otherwise, for $j = 1, \dots, k$. With this choice one has that $CB_iL_{b_i}C^{-1} = B'_iL_{b'_i}, 1 \le i \le 3$, hence $\Gamma$ and $\Gamma'$ are isomorphic, as asserted. This completes the proof of Lemma 3.4. □

The purpose of the next theorem is to estimate $b_n$, the number of isomorphism classes of groups in $\mathcal{B}_n$, and to compare it with $c_n$, the cardinal of Cobb's family $\mathcal{C}_n$.

**Theorem 3.5.** *For each dimension $n \ge 3$,*

$$\begin{aligned} b_n = \big|\big\{ &((m_1, m - m_1 + 2k_1); (m_2, m - m_2 + 2k_2); (m_3, m - m_3 + 2k_3)), \\ &\text{up to permutation } : 1 \le m_1 \le m_2 \le m_3; \\ &m_1 + m_2 + m_3 + 2(k_1 + k_2 + k_3) = n;\ m_i, k_i \in \mathbf{N}_0 \big\}\big|. \end{aligned}$$

*Furthermore, $b_n \sim B\,n^5$ and $c_n \sim C\,n^2$ as $n \to \infty$, with $B = \frac{1}{2^7 3^2 5}$ and $C = \frac{1}{2^2 3}$.*

*Proof.* By Lemma 3.4, it is clear that for each fixed dimension $n \ge 3$, there are $b_n$ isomorphism classes of Bieberbach groups in $\mathcal{B}_n$.

Since $c_m = \big|\{(m_1, m_2, m_3) : 1 \le m_1 \le m_2 \le m_3 \text{ and } m_1 + m_2 + m_3 = m\}\big|$ then $c_m = p_3(m)$, i.e. the number of partitions of $m$ with exactly three parts. It is known that

$$\frac{(m-1)(m-2)}{2^2 3} \le p_3(m) \le \frac{(m+2)(m+1)}{2^2 3}$$

(see [An], p.56). Hence, it follows that $c_m \sim \frac{1}{2^2 3} m^2$ as $m \to \infty$.

We show next that $b_n \sim Bn^5$ as $n \to \infty$.

Let $\Gamma \in \mathcal{B}_n$. Then $\Gamma \in \mathcal{B}_{n,k}$ for some $0 \le k \le \left[\frac{n-3}{2}\right]$. If we denote by

$$\begin{aligned} \mathcal{S}_{n,k} = \big\{ &\big((m_1, m - m_1 + 2k_1); (m_2, m - m_2 + 2k_2); (m_3, m - m_3 + 2k_3)\big), \\ &\text{up to permutation} : 1 \le m_1 \le m_2 \le m_3;\ m_1 + m_2 + m_3 = m; \\ &k_1 + k_2 + k_3 = k;\ m + 2k = n,\ m_i, k_i \in \mathbf{N}_0 \big\}, \end{aligned}$$

then the number of isomorphism classes of groups in $\mathcal{B}_{n,k}$ is $|\mathcal{S}_{n,k}|$ and $b_n = \sum_{k=0}^{\left[\frac{n-3}{2}\right]} |\mathcal{S}_{n,k}|$.

To obtain a lower bound of $|\mathcal{S}_{n,k}|$, we will restrict to the elements in $\mathcal{S}_{n,k}$ satisfying $m_1 < m_2 < m_3$.

Given $(m_1, m_2, m_3)$, $m_1 < m_2 < m_3$, then any two different ordered triples $(k_1, k_2, k_3)$ with $k_1 + k_2 + k_3 = k$ produce different elements in $\mathcal{S}_{n,k}$. Set $q_3(m) = \big|\{(m_1, m_2, m_3) : m_1 < m_2 < m_3 \text{ and } m_1 + m_2 + m_3 = m,\ m_i \in \mathbf{N}\}\big|$. There are $\frac{(k+1)(k+2)}{2}$ ordered triples $(k_1, k_2, k_3)$ and $q_3(m)$ triples $(m_1, m_2, m_3)$. Thus one has, at least, $\frac{(k+1)(k+2)}{2} q_3(m)$ different elements in $\mathcal{S}_{n,k}$. Since $q_3(m) \ge \frac{(m-4)(m-5)}{2^2 3}$ (see

[An], p.56) then

$$b_n \geq \sum_{k=0}^{\left[\frac{n-3}{2}\right]} \frac{(m-4)(m-5)}{2^2 3} \frac{(k+1)(k+2)}{2}$$
$$= \frac{1}{2^3 3} \sum_{k=0}^{\left[\frac{n-3}{2}\right]} (n-2k-4)(n-2k-5)(k+1)(k+2).$$

Expanding the terms of the sum, evaluating $\sum_{k=0}^{\left[\frac{n-3}{2}\right]} k^p$ for $p = 1, 2, 3, 4$ and replacing $\left[\frac{n-3}{2}\right]$ by $\frac{(n-4)}{2}$, it is easy to see that $b_n$ is greater than a polynomial in $n$ of the fifth degree, whose leading coefficient is $\frac{1}{2^7 3^2 5}$.

On the other hand,

$$b_n \leq \sum_{k=0}^{\left[\frac{n-3}{2}\right]} c_m \frac{(k+1)(k+2)}{2},$$

since for every triple $(m_1, m_2, m_3)$ with $1 \leq m_1 \leq m_2 \leq m_3$ and $m_1 + m_2 + m_3 = m$ there are at most $\frac{(k+1)(k+2)}{2}$ different elements in $\mathcal{S}_{n,k}$. Hence,

$$b_n \leq \sum_{k=0}^{\left[\frac{n-3}{2}\right]} \frac{(m+2)(m+1)}{2^2 3} \frac{(k+1)(k+2)}{2}$$
$$= \frac{1}{2^3 3} \sum_{k=0}^{\left[\frac{n-3}{2}\right]} (n-2k+2)(n-2k+1)(k+1)(k+2).$$

As before, one can show that the last expression is less than or equal to a polynomial of the fifth degree with the same leading coefficient, $\frac{1}{2^7 3^2 5}$. □

## 4. Remarks

With the help of a computer one can calculate $b_n$ and $c_n$, and compare them for small values of $n$.

To do this it is convenient to distinguish three cases:

i) $m_1 < m_2 < m_3$,

ii) $m_1 = m_2 < m_3$ or $m_1 < m_2 = m_3$,

iii) $m_1 = m_2 = m_3$. This occurs if and only if $3 | n - 2k$.

The first case was analyzed in the proof of the previous theorem. For each triple $(m_1, m_2, m_3)$ of type ii) there are $|\{(k_1, k_2, k_3) : k_1 < k_2,\ k_1 + k_2 + k_3 = k,\ k_i \in \mathbf{N}_0\}|$ different elements in $\mathcal{S}_{n,k}$. In iii), each unordered triple produces a different element of $\mathcal{S}_{n,k}$.

The results obtained, for some $n$'s, are shown in the following table.

| $n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 20 | 30 | 40 | 50 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c_n$ | 1 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 33 | 75 | 133 | 208 | 833 |
| $b_n$ | 1 | 1 | 3 | 5 | 10 | 15 | 27 | 38 | 823 | 5471 | 21571 | 63284 | 1873365 |

As seen above, $n = 5$ is the smallest dimension in which a new manifold appears which is not in Cobb's class. Denote this manifold by $M_{5,1}$. Then $M_{5,1}$ is a non-orientable compact flat manifold with fundamental group $\Gamma = \langle \alpha, \beta, \Lambda \rangle$, where $\alpha = B_1 L_{b_1}$, $\beta = B_2 L_{b_2}$ and $\Lambda$ is the canonical lattice, with

$$B_1 = \left(\begin{array}{ccc|cc} -1 & & & & \\ & -1 & & & \\ & & 1 & & \\ \hline & & & 0 & 1 \\ & & & 1 & 0 \end{array}\right), \quad B_2 = \left(\begin{array}{ccc|cc} 1 & & & & \\ & -1 & & & \\ & & -1 & & \\ \hline & & & 0 & -1 \\ & & & -1 & 0 \end{array}\right)$$

and $b_1 = \frac{e_3}{2}$, $b_2 = \frac{e_1+e_2}{2}$.

It can be proved, as in Lemma 2.2, that for these $B_1$ and $B_2$, any possible choice of $b_1$ and $b_2$ satisfying the conditions of Proposition 2.1 gives rise to isomorphic groups, thus to diffeomorphic manifolds.

Also, for $M_{5,1}$, one computes $\mathrm{H}_1(M_{5,1}, \mathbf{Z}) = \mathbf{Z}_4^2 \oplus \mathbf{Z}_2$, hence $\beta_1 = 0$, and by [Hi] one can show that $\beta_2 = 2$, $\beta_3 = 4$, $\beta_4 = 1$ and $\beta_5 = 0$.

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