PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 8, August 1996

SEQUENTIAL GROUP TOPOLOGY ON RATIONALS WITH INTERMEDIATE SEQUENTIAL ORDER

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(Communicated by Franklin D. Tall)

ABSTRACT. Using **CH** we construct a countable sequential topological group whose sequential order is between 2 and ω giving a consistent negative answer to P. Niykos' question.

1. Introduction

A space X is called *Fréchet* if $x \in \overline{A} \subseteq X$ implies the existence of a sequence in A converging to x, where $\overline{A}$ denotes the closure of A. Thus taking closure in a Fréchet space is simply adding the limit points for all sequences in a given set. So it is enough to know all the convergent sequences in a Fréchet space to reconstruct its topology. The last feature of Fréchet spaces may be given a form of exact definition. Namely a space X is *sequential* iff every nonclosed subset $A \subseteq X$ contains a sequence converging to a point outside A. Merely adding the limit points to converging sequences in a given set may not give its closure in a sequential space, but being iterated this operation gives the desired result. The minimal number of steps necessary to obtain the closure of any subset is an ordinal invariant known as the *sequential order* of the space. Here is the precise definition. Define $[A]^{seq} = \{\, x \mid x \textit{ is the limit point of a sequence in } A \,\}$. Then $[A]_0 = A$, $[A]_{\alpha+1} = [[A]_\alpha]^{seq}$, $[A]_\alpha = \bigcup_{\beta<\alpha}[A]_\beta$ for limit α. If $[A]_\alpha = \overline{A}$ for any $A \subseteq X$, then we say that the sequential order of a space X is less than or equal to α and write $so(X) \leq \alpha$. Obviously $so(X) \leq \omega_1$ for any sequential X. Now $so(X) = \min\{\, \alpha \mid so(X) \leq \alpha \,\}$. So Fréchet spaces are exactly the sequential spaces of sequential order 1.

Sequentiality and Fréchetness in topological groups were studied in several papers (see [N], [NST1], [NST2] and bibliography there). P. Nyikos in [N, Problem 4] asks whether the sequential order of a sequential topological group is ω_1 if the group is not Fréchet. In a number of papers there were constructed sequential spaces of any sequential order $\alpha < \omega_1$ which were close to topological groups. For example, L. Foged in [F] constructed for any $\alpha < \omega_1$ a regular sequential, weakly first-countable homogeneous space X_α of sequential order α. Papers [DP] and [P] contain constructions of Hausdorff sequential topological semigroups (groups in which the group operation is continuous; [DP]) and regular sequential semitopological groups ([P]; semitopological means separate continuity of the group operation and division) of any sequential order $\alpha < \omega_1$. In Example 2.12 we give a consistent negative

Received by the editors February 24, 1995.
1991 *Mathematics Subject Classification.* Primary 54D55, 54A20.
Key words and phrases. Topological group, sequential space, sequential order, Fréchet space.

answer to P. Nyikos' question by constructing a sequential topological group of countable sequential order which is not Fréchet.

Let us come to the terminology. A set $\sigma \subseteq \omega^2$ will be called *short* if $\sigma \subseteq \{0, \dots, n\} \times \omega$ for some $n \in \omega$. Let $\mathbb{Q}$ be the set of rationals. Let $\mathcal{K} = \{K_\alpha\}_{\alpha \in A}$ be an arbitrary family of subsets of $\mathbb{Q}$. Suppose $\vec{a} \in \mathbb{Q}^n$, $\vec{K} \in \mathcal{K}^n$, $n \in \omega \setminus \{0\}$. Let us denote $\langle \vec{a}, \vec{K} \rangle = \langle (a_1, \dots, a_n), (K_{\alpha_1}, \dots, K_{\alpha_n}) \rangle = a_1 \cdot K_{\alpha_1} + \dots + a_n \cdot K_{\alpha_n} \subseteq \mathbb{Q}$, where $a_i \in \mathbb{Q}$. Define $\mathbb{Q}^\infty = \bigcup_{n \in \omega} \mathbb{Q}^n$, $\mathbb{Q}^0 = \{0\}$. If $K \subseteq \mathbb{Q}$, $\vec{a} \in \mathbb{Q}^n$, we denote $\vec{a}\langle K \rangle = a_1 \cdot K + \dots + a_n \cdot K$. If $\vec{a} \in \mathbb{Q}^0$, then $\vec{a}\langle K \rangle = 0$. Let $\mathbb{Q} = \{ b_i \mid i \in \omega \}$, $b_i \neq b_j$ if $i \neq j$, $\mathbb{Q}(i) = \{ b_j \mid j \leq i \}$ and $\mathbb{Q}_k = \bigcup_{i,j \leq k} (\mathbb{Q}(i))^j$. If $a \in \mathbb{Q} \setminus \{0\}$, let $n_{\mathbb{Q}}(a) = n$ provided $a = b_n$, $n_{\mathbb{Q}}(0) = \infty > k$ for any $k \in \omega$. Let X be a scattered space (that is every subspace $F \subseteq X$ has an isolated point in F). Set $X^0 = X$, $X^{\alpha+1} = X^\alpha \setminus \{ x \mid x \textit{ is isolated in } X^\alpha \}$ and $X^\alpha = \bigcap_{\beta < \alpha} X^\beta$ for limit α. We define $Cd(X) = \min\{ \alpha \mid X^\alpha \textit{ is empty} \}$. A family γ of subsets of a topological space X is called a *k-network* if whenever $K \subseteq U$ is compact and $U \subseteq X$ is open there exists a finite $\gamma_K \subseteq \gamma$ such that $K \subseteq \bigcup \gamma_K \subseteq U$. A space X with a countable k-network is called an *$\aleph_0$-space* (see [M1], [GMT]). A quotient image of a topological sum of countably many compact spaces is called a *k_ω-space*. Every countable k_ω-space is a sequential $\aleph_0$-space, and a product of two k_ω-spaces is itself a k_ω-space (see [M2]). Consider the set $S = \omega^2 \cup \omega \cup \{\omega\}$. Define a topology on S as follows. Every point of ω^2 is isolated, a typical neighborhood of n is $\{n\} \cup (\{n\} \times \omega \setminus \textit{finitely many points})$, and $U \ni \{\omega\}$ is open if $(U \cap \{n\} \times \omega) \cup \{n\}$ is a neighborhood of n for every $n \in U$ and $\omega \setminus U$ is finite. The resulting space is called *Arens' space* S_2. All spaces are assumed to be Hausdorff.

Lemma 1.1. *A countable nondiscrete sequential topological group contains a closed copy of S_2 provided that it is a k_ω-space.*

Proof. Let G be a countable nondiscrete sequential topological group such that G is a k_ω-space. If G is not Fréchet, then it contains a closed copy of S_2 by [R]. Suppose it is Fréchet. Then being an $\aleph_0$-space, it is metrizable by [A1]. Being nondiscrete, it does not contain an isolated point. So by the well known characterization of the rationals G is homeomorphic to $\mathbb{Q}$. But $\mathbb{Q}$ is not a k_ω-space (see [M2]). A contradiction. □

2. General construction of a group topology on $\mathbb{Q}$

In this section we provide a general construction of Hausdorff group topologies on $\mathbb{Q}$; the construction will be split into a sequence of technical lemmas.

Lemma 2.1. *Let $\mathcal{K} = \{K_n\}_{n \in \omega}$ be an arbitrary family of subsets of $\mathbb{Q}$. Then there exists a countable family $C(\mathcal{K}) \supseteq \mathcal{K}$ such that:*

(a) *$\{a\} \in C(\mathcal{K})$ for any $a \in \mathbb{Q}$,*
(b) *if $\vec{a} \in \mathbb{Q}^n$, $\vec{K} \in C(\mathcal{K})^n$, then $\langle \vec{a}, \vec{K} \rangle \in C(\mathcal{K})$,*
(c) *if $K^1 \in C(\mathcal{K})$, ... , $K^n \in C(\mathcal{K})$, then $\bigcup_{i \leq n} K^i \in C(\mathcal{K})$,*
(d) *if $\mathcal{K} \subseteq C'(\mathcal{K})$ and $C'(\mathcal{K})$ has properties* (a)–(c)*, then $C(\mathcal{K}) \subseteq C'(\mathcal{K})$.*

Proof. Put $c\mathcal{K} = \mathcal{K} \cup \{ \{a\} \mid a \in \mathbb{Q} \}$ and $C(\mathcal{K}) = \{ \bigcup_{i \leq k} \langle \vec{a}_i, \vec{K}_i \rangle \mid \vec{a}_i \in \mathbb{Q}^{n_i}, \vec{K}_i \in c\mathcal{K}^{n_i}, n_i \in \omega, k \in \omega \}$. □

Lemma 2.2. *If $\mathcal{K} = \{K_n\}_{n \in \omega}$ is a family of compact subsets of $\mathbb{Q}$, then $C(\mathcal{K})$ is also a family of compact subsets of $\mathbb{Q}$.*

Lemma 2.3. *Let $\mathcal{K} = \{K_n\}_{n\in\omega}$ be an arbitrary family of compact subsets of $\mathbb{Q}$. Let us introduce a new topology on $\mathbb{Q}$. We make $U \subseteq \mathbb{Q}$ open if and only if $U \cap F$ is relatively open for every $F \in C(\mathcal{K})$. Denote $\mathbb{Q}$ with this topology by $G(\mathcal{K})$. Then:*

(e) *if $\vec{a} \in \mathbb{Q}^n$, then the mapping $p : G(\mathcal{K})^n \to G(\mathcal{K})$, $p(\vec{b}) = \langle \vec{a}, \vec{b}\rangle$ is continuous,*
(f) *$G(\mathcal{K})$ is a topological group,*
(g) *$G(\mathcal{K})$ is a k_ω-space.*

Proof. Let $C(\mathcal{K}) = \{F_i\}_{i\in\omega}$. Let $\vec{a} \in \mathbb{Q}^n$. Without loss of generality we may assume that $F_0 = \{0\}$. To prove (e) it is enough to show that for any $U \ni 0$, open in $G(\mathcal{K})$, there are $V_1, \dots, V_n$ open in $G(\mathcal{K})$, such that $\langle \vec{a}, (V_1, \dots, V_n)\rangle \subseteq U$ and $0 \in \bigcap_{k\le n} V_k$. Let us construct a family $\{V_k^i\}_{k\le n, i\in\omega}$ so that:

$$0 \in V_k^i, \tag{1}$$

$$V_k^i \text{ is a relatively open subset of } \{0\} \cup \bigcup_{j<i} F_j, \tag{2}$$

$$\langle \vec{a}, (\overline{V_1^i}, \dots, \overline{V_n^i})\rangle \subseteq U, \tag{3}$$

$$\overline{V_k^i} \subseteq V_k^{i+1}. \tag{4}$$

Put $V_k^0 = F_0 = \{0\}$. Suppose we have constructed V_k^i satisfying (1)–(4) for all $k \le n$, $i \le m$. Let us construct V_k^{m+1} for $k \le n$. Consider compact sets $F'_k = \{0\} \cup \bigcup_{i<m+1} F_i$. Then $F = \langle \vec{a}, (F'_1, \dots, F'_n)\rangle \in C(\mathcal{K})$ by (b) and thus $U \cap F$ is relatively open. Hence there exists U' open in the usual topology of $\mathbb{Q}$ such that $U' \cap F = U \cap F$. Since $V_k^m \subseteq \{0\} \cup \bigcup_{i<m} F_i$, it follows that $\langle \vec{a}, (\overline{V_1^m}, \dots, \overline{V_n^m})\rangle \subseteq U \cap F = U' \cap F \subseteq U'$. Now $\overline{V_k^m}$ are compact subsets of $\mathbb{Q}$ in the usual topology. Then it is easy to find V'_k open in the usual topology of $\mathbb{Q}$ so that $\overline{V_k^m} \subseteq V'_k$ and $\langle \vec{a}, (\overline{V'_1}, \dots, \overline{V'_n})\rangle \subseteq U'$. Now put $V_k^{m+1} = V'_k \cap F'_k$, and (1)–(4) are easy to check. Set $V_k = \bigcup_{i\in\omega} V_k^i$. It is a routine procedure to check that V_k is as required. Now (f) easily follows from (e); and (g) follows from the fact that the obvious mapping $p : \oplus C(\mathcal{K}) \to G(\mathcal{K})$ is quotient, where $\oplus$ denotes the topological sum. □

Some properties of $G(\mathcal{K})$ and $C(\mathcal{K})$ are discussed in the next three lemmas.

Lemma 2.4. *If $\mathcal{K} = \bigcup_{\beta<\alpha} \mathcal{K}_\beta$ and $\mathcal{K}_\beta \subseteq \mathcal{K}_{\beta'}$ for $\beta \le \beta'$, then $C(\mathcal{K}) = \bigcup_{\beta<\alpha} C(\mathcal{K}_\beta)$.*

Lemma 2.5. *Let $\mathcal{K} = \{K_i\}_{i\in\omega}$ be a family of compact subsets of $\mathbb{Q}$, and $\mathcal{U} = \{U_i\}_{i\in\omega}$ a family of open subsets of $G(\mathcal{K})$. Then there exists a topology $\tau(\mathcal{U}, \mathcal{K})$ such that:*

(h) *the mapping $p : \mathbb{Q}^n \to \mathbb{Q}$, where $p(\vec{a}) = \langle \vec{b}, \vec{a}\rangle$, $\vec{b} \in \mathbb{Q}^n$, is continuous in $\tau(\mathcal{U}, \mathcal{K})$,*
(i) *$\tau(\mathcal{U}, \mathcal{K})$ is a Hausdorff group topology with a countable base,*
(j) *$U_i \in \tau(\mathcal{U}, \mathcal{K})$ for any $n \in \omega$,*
(k) *$\tau(\mathcal{U}, \mathcal{K})$ is stronger than the usual topology of $\mathbb{Q}$ and weaker than the topology of $G(\mathcal{K})$, and*
(l) *if $\mathcal{U} \supseteq \tau_0(\mathcal{U}', \mathcal{K}')$, then $\tau(\mathcal{U}, \mathcal{K})$ is stronger than $\tau(\mathcal{U}', \mathcal{K}')$, where $\mathcal{K}$ and $\mathcal{K}'$ are countable families of compact subsets of $\mathbb{Q}$ and $\tau_0(\mathcal{U}, \mathcal{K})$ is a fixed countable base at $0 \in \mathbb{Q}$ in $\tau(\mathcal{U}, \mathcal{K})$.*

Proof. Without loss of generality we may assume that $\mathcal{U}$ contains a countable base for $\mathbb{Q}$ in the usual topology. Let $\mathcal{M} = \{M_i\}_{i\in\omega}$ be an arbitrary family of subsets of $\mathbb{Q}$ open in $G(\mathcal{K})$. Consider the following operations on $\mathcal{M}$:

$$C_1(\mathcal{M}) = \{\, N \mid N \textit{ is a finite intersection of sets from } \mathcal{M} \,\},$$
$$C_2(\mathcal{M}) = \{\, N \mid N \textit{ is of the form } a + M_i \textit{ where } a \in \mathbb{Q}, M_i \in \mathcal{M} \,\}.$$

Since for every $\vec{b} \in \mathbb{Q}^n$ the mapping $p : \mathbb{Q}^n \to \mathbb{Q}$, where $p(\vec{a}) = \langle \vec{b}, \vec{a} \rangle$, is continuous in the topology of $G(\mathcal{K})$ by (e), it follows that for any $M_i \in \mathcal{M}$ any $\vec{a} \in \mathbb{Q}^n$, $\vec{b} \in \mathbb{Q}^n$ such that $\langle \vec{a}, \vec{b} \rangle \in M_i$, $\vec{a} = (a_1, \ldots, a_n)$ there are $O_1(\vec{a}, \vec{b}, M_i) \ni a_1, \ldots, O_n(\vec{a}, \vec{b}, M_i) \ni a_n$ such that $\langle \vec{b}, (O_1(\vec{a}, \vec{b}, M_i), \ldots, O_n(\vec{a}, \vec{b}, M_i)) \rangle \subseteq M_i$ and $O_j(\vec{a}, \vec{b}, M_i)$ is open in $G(\mathcal{K})$. Define

$$C_3(\mathcal{M}) = \{\, O_i(\vec{a}, \vec{b}, M_j) \mid \vec{a}, \vec{b} \in \mathbb{Q}^n, i \le n, n \in \omega, M_j \in \mathcal{M}, \langle \vec{a}, \vec{b} \rangle \in M_j \,\} \cup \mathcal{M}.$$

It is obvious that all the operations produce countable families of open subsets of $G(\mathcal{K})$. Now put $\mathcal{M}_0 = \mathcal{U}$, $\mathcal{M}_1 = C_1(\mathcal{M}_0)$, $\mathcal{M}_2 = C_2(\mathcal{M}_1)$, $\mathcal{M}_3 = C_3(\mathcal{M}_2)$ and $\mathcal{M}_{3n+1} = C_1(\mathcal{M}_{3n})$, $\mathcal{M}_{3n+2} = C_2(\mathcal{M}_{3n+1})$, $\mathcal{M}_{3(n+1)} = C_3(\mathcal{M}_{3n+2})$. Now define $\mathcal{N} = \bigcup_{n \in \omega} \mathcal{M}_n$. Then $\mathcal{N}$ is a countable family of open in $G(\mathcal{K})$ sets. Take it as a base for the topology $\tau(\mathcal{U}, \mathcal{K})$ on $\mathbb{Q}$. Then (h)–(l) are easy to check. □

From now on we fix, for every pair $(\mathcal{U}, \mathcal{K})$ satisfying the conditions of Lemma 2.5, a topology $\tau(\mathcal{U}, \mathcal{K})$ having properties (h)–(l).

Lemma 2.6. $C(\mathcal{K} \cup \{K\}) = \{\, \bigcup_{i \le k} (\vec{a}_i \langle K \rangle + K^i) \mid \vec{a}_i \in \mathbb{Q}^\infty, K^i \in C(\mathcal{K}), k \in \omega \,\}$.

Proof. Let us denote the right side of the equality by C'. We show that C' has properties (a)–(c). Then by (d) we get what is required.

Properties (a) and (c) are obvious. To prove (b) it is enough to check that if $\vec{a} \in \mathbb{Q}^n$, $\vec{F} = (\vec{a}_1 \langle K \rangle + K^1, \ldots, \vec{a}_n \langle K \rangle + K^n)$, then $\langle \vec{a}, \vec{F} \rangle \in C'$. If $\vec{a} = (a_1, \ldots, a_n)$, then

$$\langle \vec{a}, \vec{F} \rangle = a_1 \cdot \vec{a}_1 \langle K \rangle + \cdots + a_n \cdot \vec{a}_n \langle K \rangle + \langle \vec{a}, (K^1, \ldots, K^n) \rangle.$$

It follows that $\langle \vec{a}, \vec{F} \rangle = \vec{a'} \langle K \rangle + K'$ for some $\vec{a'} \in \mathbb{Q}^\infty$ and $K' \in C(\mathcal{K})$. □

Let us consider an example of a group $G(\mathcal{S})$. Define $S_1 = \{\, 1/n \mid n \in \mathbb{N} \,\} \cup \{0\}$, $\mathcal{S} = \{S_1\}$ and consider the topological group $G(\mathcal{S})$. It is obviously nondiscrete and is a k_ω-space by (g). Then it contains a closed copy of S_2 by Lemma 1.1. So we can fix an injection $t : \omega^2 \to \mathbb{Q}$ such that:

(m) for any $n \in \omega$, $t(n, k) \to s_n$ as $k \to \infty$ in $G(\mathcal{S})$,
(n) $s_n \to 0$ as $n \to \infty$ in $G(\mathcal{S})$,
(o) $0 \notin t(\omega^2)$, $0 \ne s_n \ne s_k \notin t(\omega^2)$ if $n \ne k$,
(p) $t(\omega^2) \cup \{\, s_i \mid i \in \omega \,\} \cup \{0\}$ is a closed subset of $G(\mathcal{S})$ homeomorphic to S_2.

The following property is easy to satisfy using first countability of $\mathbb{Q}$, the definition of $G(\mathcal{S})$ and properties (m), (n).

(q) $\overline{t(\omega^2)}^{\mathbb{Q}}$ is a compact subset of $\mathbb{Q}$ and $Cd(\overline{t(\omega^2)}^{\mathbb{Q}})$ is finite.

Properties (m)–(p) imply that $t^{-1}(F)$ is short for any $F \in C(\mathcal{S})$. In all further considerations t denotes the injection discussed above.

Remark. It may be shown that the construction of $G(\mathcal{K})$ presented here could be carried over with minor changes to any Abelian group G instead of $\mathbb{Q}$, and in the case when $\mathcal{K}$ consists of a single convergent sequence the topology of $G(\mathcal{K})$ coincides with that of $G\{\varphi\}$ from [ZP], where φ is the filter corresponding to the sequence.

The author is very obliged to Professor D. Shakhmatov who called the author's attention to the paper [ZP].

The following lemma is well known.

Lemma 2.7. *Assume that K_0 and K_1 are compact sets, $K = K_0 \cup K_1$ and $Cd(K_\nu)$ is finite for $\nu = 0, 1$. Then $Cd(K_0 \times K_1)$ is finite, and $Cd(f(K))$ is finite for an arbitrary continuous f.*

One can easily prove the next lemma using the definition of $C(\mathcal{K})$ and Lemma 2.7.

Lemma 2.8. *If $\mathcal{K} = \{K_i\}_{i\in\omega}$ and $Cd(K_i)$ is finite for any $i \in \omega$, then $Cd(K)$ is finite for any $K \in C(\mathcal{K})$.*

Let τ be any topology on $\mathbb{Q}$. Let us say that a set $A \subseteq \mathbb{Q}$ has *ω-property in τ* if there is a compact subspace $K \ni 0$ of $(\mathbb{Q}, \tau)$ such that:
(r) $Cd(K) = \omega + 1$,
(s) $Cd(K \setminus U)$ is finite for any $U \ni 0$ open in K, and
(t) $K \setminus K^1 \subseteq A$.

The next lemma has a routine proof which is omitted.

Lemma 2.9. *Let τ_1 be a topology on $\mathbb{Q}$ such that $(\mathbb{Q}, \tau_1)$ is a sequential space and τ_1 is stronger than some first-countable topology τ_2 on $\mathbb{Q}$. If $0 \in [A]^{\tau_1}_{\omega+1}$ and $0 \notin [A]^{\tau_1}_{\omega}$ for some $A \subseteq \mathbb{Q}$, then A has ω-property in τ_2.*

Lemma 2.10. *Let τ be a first-countable topology on $\mathbb{Q}$ stronger than the usual topology of $\mathbb{Q}$, K be a compact subspace of $(\mathbb{Q}, \tau)$ and $\mathcal{K} = \{K_i\}_{i\in\omega}$ be a family of compact subsets of $\mathbb{Q}$ such that*

$$Cd(K) = \omega + 1, \tag{5}$$

$$Cd(K \setminus U) \text{ is finite for any } U \ni 0 \text{ open in } K, \tag{6}$$

$$Cd(K_n) \text{ is finite for any } n \in \omega. \tag{7}$$

Suppose also that the injection $t : \omega^2 \to \mathbb{Q}$ has the following property:

$$\text{for any } F \in C(\mathcal{K}) \text{ the set } t^{-1}(F) \text{ is short.} \tag{8}$$

Then there exists a sequence $S = \{ p_i \mid i > 0 \} \subseteq K \setminus K^1$ such that:
(u) *$p_i \to 0$ as $i \to \infty$ in τ,*
(v) *for any $\vec{a} \in \mathbb{Q}^\infty$ any $F \in C(\mathcal{K})$ the set $t^{-1}(\vec{a}\langle S \cup \{0\}\rangle + F)$ is short.*

Proof. Let $C(\mathcal{K}) = \{F_i\}_{i\in\omega}$. First fix $\{U_i\}_{i\in\omega}$, a base of open and closed neighborhoods of the point $0 \in K$, where the topology on K is induced from $(\mathbb{Q}, \tau)$; obviously such topology coincides with that induced from the usual one of $\mathbb{Q}$, so we will not mention the exact topology in our considerations. Put $p_0 = 0$. Now we choose points p_k, $k \geq 1$ by induction so that:

$$p_k \notin \bigcup_{\substack{n_{\mathbb{Q}}(a) \leq k, \\ \vec{a} \in \mathbb{Q}_k}} \Big(\overline{t(\omega^2)}^{\mathbb{Q}} - \Big(\bigcup_{i \leq k} F_i + \vec{a}\langle \{ p_i \mid i < k \} \rangle \Big) \Big) \cdot a^{-1}, \tag{9}$$

$$p_k \in U_k \cap (K \setminus K^1). \tag{10}$$

Let us denote the union in the right side of (9) by M_k. Since $\overline{t(\omega^2)}^{\mathbb{Q}}$ is compact and $Cd(\overline{t(\omega^2)}^{\mathbb{Q}})$ is finite (see (q)), it follows from (7), Lemma 2.7 and Lemma 2.8 that the set M_k is also compact and $Cd(M_k)$ is finite. By the choice of $\{U_i\}_{i\in\omega}$ the sets $U_k\cap K$ and $K\setminus U_n$ are compact. By Lemma 2.7 and (6), we have $Cd(U_k\cap K)=\omega+1$. Then there exists a point $p_k\in U_k\cap(K\setminus K^1)$ such that $p_k\notin M_k$. Now (10) implies (u).

Now consider the set $R=\vec{a}\langle S\cup\{0\}\rangle+F_n$ for some $n\in\omega$. We have $\vec{a}=(a_1,\dots,a_k)$ for some $k\in\omega$. So $\vec{a}\in\mathbb{Q}_{i(\vec{a})}$ for some $i(\vec{a})\in\omega$. The set $A=\{\,\langle\vec{a},\vec{b}\rangle\mid\vec{b}\in\{0,1\}^k\,\}\setminus\{0\}$ is finite, so there exists $r=\max\{\,n_{\mathbb{Q}}(a)\mid a\in A\,\}<\infty$. Let $M=\max\{i(\vec{a}),r,n\}$. Then

$$R=\bigcup_{(i_1,\dots,i_k)\in\omega^k} a_1\cdot p_{i_1}+\dots+a_k\cdot p_{i_k}+F_n.$$

We write $i\in_e(i_1,\dots,i_k)$ if and only if $\sum_{i_\nu=i}a_\nu\neq 0$ or $p_i=0$. It is easy to see that if $a_1\cdot p_{i_1}+\dots+a_k\cdot p_{i_k}=b\in\mathbb{Q}$, then there are $\{p_{j_1},\dots,p_{j_k}\}\subseteq\{p_{i_1},\dots,p_{i_k}\}\cup\{p_0\}$ such that $a_1\cdot p_{j_1}+\dots+a_k\cdot p_{j_k}=b$ and $j_m\in_e(j_1,\dots,j_k)$ for any $m\leq k$. A point $(i_1,\dots,i_k)\in\omega^k$ is called *essential* if $i_m\in_e(i_1,\dots,i_k)$ for any $m\leq k$. Let $\Omega\subseteq\omega^k$ be the set of all the essential points. Using the properties of essential points discussed above, it is easy to check

$$R=\bigcup_{(i_1,\dots,i_k)\in\Omega} a_1\cdot p_{i_1}+\dots+a_k\cdot p_{i_k}+F_n.$$

Let us denote

$$L=\bigcup_{(i_1,\dots,i_k)\in\Omega\setminus\{\,i\mid i\leq M\,\}^k} a_1\cdot p_{i_1}+\dots+a_k\cdot p_{i_k}+F_n.$$

Obviously

$$\vec{a}\langle S\cup\{0\}\rangle+F_n=R=(\vec{a}\langle\{\,p_i\mid i\leq M\,\}\rangle+F_n)\cup L. \tag{11}$$

Let us prove that $t(\omega^2)\cap L=\varnothing$. Suppose the contrary. Then there is a point $t\in t(\omega^2)$ such that

$$t=a_1\cdot p_{i_1}+\dots+a_k\cdot p_{i_k}+f, \tag{12}$$

where $(i_1,\dots,i_k)\in\Omega\setminus\{\,i\mid i\leq M\,\}^k$ and $f\in F_n$. Without loss of generality assume that $i_1=\max\{\,i_j\mid j\leq k\,\}$. Note that $i_1>M$. Substituting every occurrence of p_{i_1} in (12) by $p_0=0$, and leaving the occurrences of other p_i's untouched (and thus obtaining p_{j_i}'s) we have

$$\Big(\sum_{i_\nu=i_1}a_\nu\Big)\cdot p_{i_1}=t-(a_1\cdot p_{j_1}+\dots+a_k\cdot p_{j_k}+f),$$

where $j_i<i_1$ for $i\leq k$. Thus $a=\sum_{i_\nu=i_1}a_\nu\neq 0$ (because $(i_1,\dots,i_k)\in\Omega$) and

$$p_{i_1}=(t-(f+a_1\cdot p_{j_1}+\dots+a_k\cdot p_{j_k}))\cdot a^{-1},$$

where $\vec{a}\in\mathbb{Q}_M$, $M<i_1$, $n_{\mathbb{Q}}(a)\leq r\leq M<i_1$, $f\in F_n$ and $n\leq M<i_1$, which contradicts (9).

Now $(\vec{a}\langle\{\,p_i\mid i\leq M\,\}\rangle+F_n)\in C(\mathcal{K})$, so we get (v) by (11) and (8). □

Let **CH** hold. Let $\{O_\alpha\}_{\alpha<\omega_1}$ be all subsets of $\mathbb{Q}$. For technical purposes it is convenient to require that $O_0=\varnothing$. It is also convenient to put $\tau(\{\varnothing\},\varnothing)$ to be the usual topology of $\mathbb{Q}$. It is easy to check that properties (h)–(l) are satisfied in this case.

Lemma 2.11 (CH). *For every $\alpha < \omega_1$ there exist*

a countable family $\mathcal{K}_\alpha$ of compact subsets of $\mathbb{Q}$,
a countable family $\mathcal{U}_\alpha$ of subsets of $\mathbb{Q}$ and
a compact subset K_α of $\mathbb{Q}$ such that:

(13) $\mathcal{K}_\alpha = \mathcal{K}_{<\alpha} \cup \{K_\alpha\}$, *where* $\mathcal{K}_{<\alpha} = \bigcup_{\beta<\alpha} \mathcal{K}_\beta$ *and* $S_1 \in \mathcal{K}_\alpha$,
(14) *if O_α has ω-property in $\tau(\mathcal{U}_\alpha, \mathcal{K}_{<\alpha})$, then $K_\alpha \subseteq O_\alpha \cup \{0\}$ is a nontrivial convergent sequence in $\mathbb{Q}$ (and thus in $G(\mathcal{K}_\alpha)$) with the limit point 0; otherwise $K_\alpha = S_1$,*
(15) *if $U \in \tau(\mathcal{U}_\beta, \mathcal{K}_{<\beta})$, $\beta \le \alpha$, then U is open in $G(\mathcal{K}_\alpha)$,*
(16) $\mathcal{U}_\alpha \supseteq \bigcup_{\beta<\alpha} \tau_0(\mathcal{U}_\beta, \mathcal{K}_{<\beta})$,
(17) *for every $\beta \le \alpha$ the topology of $G(\mathcal{K}_\alpha)$ is stronger than $\tau(\mathcal{U}_\beta, \mathcal{K}_{<\beta})$,*
(18) *if O_α is open in $G(\mathcal{K}_\alpha)$, then $O_\alpha \in \mathcal{U}_\alpha$,*
(19) *for every $F \in C(\mathcal{K}_\beta)$, $\beta \le \alpha$ the set $t^{-1}(F)$ is short.*

Proof. Set $\mathcal{K}_0 = \{S_1\}$, $K_0 = S_1$ and $\mathcal{U}_0 = \{\varnothing\}$. Then (13)–(19) are easy to check using the fact that $O_0 = \varnothing$, $\tau(\{\varnothing\}, \varnothing)$ is the usual topology of $\mathbb{Q}$, (k), and (m)–(p). Suppose we have already constructed $\mathcal{K}_\alpha$, K_α, $\mathcal{U}_\alpha$ so that they satisfy (13)–(19) for all $\alpha < \kappa$. Define

$$\mathcal{U}_{(1)} = \bigcup_{\alpha<\kappa} \tau_0(\mathcal{U}_\alpha, \mathcal{K}_{<\alpha}), \quad \mathcal{K}_{<\kappa} = \bigcup_{\alpha<\kappa} \mathcal{K}_\alpha. \tag{20}$$

If O_κ is open in $G(\mathcal{K}_{<\kappa})$, put

$$\mathcal{U}_\kappa = \mathcal{U}_{(1)} \cup \{O_\kappa\}. \tag{21}$$

Otherwise put $\mathcal{U}_\kappa = \mathcal{U}_{(1)}$. It follows from the induction, (20) and (17) that if $U \in \mathcal{U}_{(1)}$, then U is open in $G(\mathcal{K}_\alpha)$ provided $\alpha \ge \beta$ for some $\beta < \omega_1$. Since obviously the topology of $G(\mathcal{K}_\alpha)$ is stronger than the topology of $G(\mathcal{K}_\beta)$ when $\alpha < \beta$, it follows that U is open in $G(\mathcal{K}_\alpha)$ for any $\alpha < \kappa$. So $U \cap F$ is relatively open for every $F \in C(\mathcal{K}_\alpha)$, $\alpha < \kappa$. Now it easily follows from (13) that $\mathcal{K}_\beta \subseteq \mathcal{K}_\alpha$ if $\beta < \alpha$, and Lemma 2.4 gives that $U \cap F$ is relatively open for every $F \in C(\mathcal{K}_{<\kappa})$. Then U is open in $G(\mathcal{K}_{<\kappa})$. So we can consider the topology $\tau(\mathcal{U}_\kappa, \mathcal{K}_{<\kappa})$.

Suppose that O_κ has ω-property in $\tau(\mathcal{U}_\kappa, \mathcal{K}_{<\kappa})$. Then there is a compact subspace K of $(\mathbb{Q}, \tau(\mathcal{U}_\kappa, \mathcal{K}_{<\kappa}))$ satisfying conditions (r)–(t) and thus (5), (6). Property (8) holds by induction and (19); (7) follows from induction, (14) and (13). Using (t), (5)–(8) and Lemma 2.10 we choose $S = \{\, p_i \mid i > 0 \,\} \subseteq O_\kappa \cap K$ such that

$$p_i \to 0 \textit{ as } i \to \infty \textit{ in } \tau(\mathcal{U}_\kappa, \mathcal{K}_{<\kappa}), \tag{22}$$

and

$$\begin{gathered} t^{-1}(\vec{a}\langle S \cup \{0\}\rangle + F) \textit{is short} \\ \textit{for any } \vec{a} \in \mathbb{Q} \textit{ and any } F \in C(\mathcal{K}_{<\alpha}). \end{gathered} \tag{23}$$

Then we set $K_\kappa = S \cup \{0\}$, $\mathcal{K}_\kappa = \bigcup_{\alpha<\kappa} \mathcal{K}_\alpha \cup \{K_\kappa\}$. Then (13) holds. By (22) and the choice of K_κ and $\mathcal{K}_\kappa$, (14) is also satisfied.

Let $U \in \tau(\mathcal{U}_\kappa, \mathcal{K}_{<\kappa})$. Let us show that U is open in $G(\mathcal{K}_\kappa)$. To show this it is enough to prove that $U \cap F$ is relatively open (in the usual topology of $\mathbb{Q}$) for every $F \in C(\mathcal{K}_\kappa)$. By Lemma 2.6, (20) and Lemma 2.4 every F is of the form

$$F = \bigcup_{i \le k} \vec{a}_i \langle K_\kappa \rangle + F_i, \tag{24}$$

where $\vec{a}_i \in \mathbb{Q}^\infty$, $F_i \in C(\mathcal{K}_\alpha)$ for some $\alpha < \kappa$ and $k \in \omega$. Now K_κ is compact in $\tau(\mathcal{U}_\kappa, \mathcal{K}_{<\kappa})$, and by (h) every $\vec{a}_i\langle K_\kappa\rangle$ is compact in $\tau(\mathcal{U}_\kappa, \mathcal{K}_{<\kappa})$. Thus F is compact in $\tau(\mathcal{U}_\kappa, \mathcal{K}_{<\kappa})$ and has the topology induced from $\mathbb{Q}$ by (k). But U is open in $\tau(\mathcal{U}_\kappa, \mathcal{K}_{<\kappa})$ so $U \cap F$ is relatively open. Thus (15) holds, (16) follows from (20) and (18) follows from (21). Then the topology of $G(\mathcal{K}_\kappa)$ is stronger than $\tau(\mathcal{U}_\kappa, \mathcal{K}_{<\kappa})$ by (15) and $\tau(\mathcal{U}_\kappa, \mathcal{K}_{<\kappa})$ is stronger than $\tau(\mathcal{U}_\beta, \mathcal{K}_{<\beta})$ for $\beta < \kappa$ by (16) and (l). So (17) holds. Now (23) and Lemma 2.6 give (19). □

In the example below we construct a sequential topological group G such that $2 \le so(G) \le \omega$.

Example 2.12 (CH). Let $\mathcal{K} = \bigcup_{\alpha<\omega_1} C(\mathcal{K}_\alpha)$, where $\mathcal{K}_\alpha$ are countable families of compact subsets of $\mathbb{Q}$ constructed in Lemma 2.11. Let G be the set $\mathbb{Q}$ equipped with the topology defined as follows: $U \subseteq G$ is open if $U \subseteq F$ is relatively open for any $F \in \mathcal{K}$.

Fact. *For every $\beta < \omega_1$ the topology $\tau(\mathcal{U}_\beta, \mathcal{K}_{<\beta})$ is weaker than the topology of G.*

Proof of the fact. Indeed let $U \in \tau(\mathcal{U}_\beta, \mathcal{K}_{<\beta})$. Consider an arbitrary compact $K \in \mathcal{K}$. It is enough to prove that $U \cap K$ is relatively open. But $K \in C(\mathcal{K}_\alpha)$ for some $\alpha < \omega_1$. Without loss of generality we assume that $\beta \le \alpha$. Then by (17) U is open in $G(\mathcal{K}_\alpha)$ thus $U \cap K$ is relatively open. □

Obviously in the topology defined above G is a quotient space of the topological sum $\oplus\{\, F \mid F \in \mathcal{K}\,\}$. So G is sequential. Consider now an arbitrary $O \subseteq \mathbb{Q}$ open in G. Then O is listed among $\{O_\alpha\}_{\alpha\in\omega_1}$ and thus $O = O_\alpha$ for some $\alpha < \omega_1$. It follows from the definition of $\mathcal{K}$ and $G(\mathcal{K}_\alpha)$ that the topology of $G(\mathcal{K}_\alpha)$ is stronger than the topology of G. So $O = O_\alpha$ is open in $G(\mathcal{K}_\alpha)$ and by (18) and (j) $O \in \tau(\mathcal{U}_\alpha, \mathcal{K}_{<\alpha})$. Now it follows from the fact proved above that $O \subseteq \mathbb{Q}$ is open in G if and only if O is open in $\mathbb{Q}$ equipped with the topology $\tau(\mathcal{U}_\alpha, \mathcal{K}_{<\alpha})$ for some $\alpha < \omega_1$. Thus the topology of G is the upper bound for the family $\{\, \tau(\mathcal{U}_\alpha, \mathcal{K}_{<\alpha}) \mid \alpha < \omega_1\,\}$. Since every $\tau(\mathcal{U}_\alpha, \mathcal{K}_{<\alpha})$ is a group topology on $\mathbb{Q}$, the topology of G is a group topology on $\mathbb{Q}$.

Now let t be the bijection described before Lemma 2.7. Obviously the topology of G is weaker than the topology of $G(\mathcal{S})$ (see (13) and the definition of $G(\mathcal{S})$ before Lemma 2.7). Thus $0 \in \overline{t(\omega^2)}$ in G by (m) and (n). Let $\{x_i\}_{i\in\omega} \subseteq t(\omega^2)$ be such that $x_i \to 0$ as $i \to \infty$ in G. Without loss of generality by the construction of G we may assume that there is $F \in \mathcal{K}$ such that $\{x_i\}_{i\in\omega} \subseteq F$. Then from (19) it follows that $0 \in \overline{t(\{0,\dots,n\} \times \omega)}$ for some $n \in \omega$ which contradicts the choice of t because $\overline{t(\{0,\dots,n\} \times \omega)} = t(\{0,\dots,n\} \times \omega) \cup \{\, s_i \mid i \le n\,\} \not\ni 0$ (see (m), (o)). So G is not Fréchet.

Suppose that $so(G) > \omega$. Then there exists $A \subseteq G$ such that $0 \in [A]_{\omega+1}$ and $0 \notin [A]_\omega$. But $A = O_\alpha$ for some $\alpha < \omega_1$. The topology $\tau(\mathcal{U}_\alpha, \mathcal{K}_{<\alpha})$ is first-countable and it follows from the fact, Lemma 2.9 and (14) that $K_\alpha \subseteq O_\alpha \cup \{0\} = A \cup \{0\}$ and K_α is a nontrivial convergent sequence with the limit point 0 in $G(\mathcal{K}_\alpha)$. Thus K_α is a convergent sequence in G, a contradiction. Therefore, $so(G) \le \omega$. □

The author wishes to thank Professor T. Nogura for many useful suggestions, advice and moral support.

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