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FATOU THEOREMS FOR PARABOLIC EQUATIONS

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ABSTRACT. For elliptic parabolic operators with time dependent coefficients, bounded and measurable, the absolute continuity of the two caloric measures plus a Fatou theorem are shown to hold on the parabolic boundary of a smooth cylinder given a Carleson-type condition on the coefficients of the operators, and assuming one of the measures is a center doubling measure. Given a stronger Carleson condition, and no doubling assumption, another kind of Fatou theorem result holds. The method of proof follows that of Fefferman, Kenig and Pipher.

There has been much work done in the past 20 years on extending classical results for harmonic functions on bounded domains to solutions of elliptic and parabolic equations. The present paper is concerned with non-tangential convergence to boundary data of solutions to

$$\text{(A)} \qquad \begin{aligned} \left(\frac{\partial}{\partial t} - L\right) u = 0 \text{ in } D_T = D \times [0,T], \\ u\big|_{\partial_p D_T} = f \in L^2\left(\partial_p D_T\right) \end{aligned}$$

on a smooth cylinder domain D_T. L is a strictly elliptic operator with time-dependent coefficients, bounded and measurable.

The Fatou theorem for solutions to (A) with $f \in L^\infty\left(\partial_p D_T\right)$ and coefficients of L time-independent was proved by Fabes, Garofalo and Salsa in their paper "A backward Harnack inequality and Fatou theorem for nonnegative solutions of parabolic equations" [8]. A key estimate in their proof was the center doubling condition for parabolic measure (see below); with this condition it is possible to obtain non-tangential convergence of a solution at the boundary of its domain by a classical argument [12], [8]. As Fabes, Garofalo and Salsa showed the center doubling condition for a caloric measure is equivalent to a backward Harnack inequality for the Green's function at the boundary [8]. Yannick Heurteaux [10] has also obtained the center doubling condition for caloric measures whose associated operators have time-dependent coefficients which satisfy certain Lipschitz conditions. She obtains a comparison of measures as well as the Fatou theorem and backward Harnack inequality. As far as I am aware it is not known whether a caloric measure whose operator has time-dependent coefficients which are only assumed to be L^∞ satisfies a center-doubling condition or not.

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In this paper a different approach to the Fatou theorem is used. The key ingredient is a Carleson-type condition for two caloric measures; the proofs are in the spirit of the stopping time arguments in Fefferman, Kenig and Pipher "The theory of weights and the Dirichlet problem for elliptic equations" [9]. Non-tangential convergence of solutions to (A) is obtained for operators L whose coefficients are bounded and measurable, and satisfy a Carleson-type condition (C_1) and (C_2) in Theorems 1 and 2 below) with respect to a second operator M. All additional assumptions hold only for M. Consequently a Fatou theorem is obtained for solutions to operators whose associated caloric measures are not assumed to satisfy a center-doubling condition.

The center doubling condition for a caloric measure $d\omega^{(X_0 T_0)}(Q,s) = d\omega(Q,s)$ is that for all parabolic boundary disks $\Delta_r(Q_0,s_0) = \{(Q,s) \in \partial_p D_T : d_p(Q,s;Q_0,s_0) < r\}$ there is a constant C independent of (Q_0,s_0) and r so that

$$\omega\left(\Delta_r\left(Q_0,s_0\right)\right) \leq C\omega(\Delta_{\frac{r}{2}}\left(Q_0,s_0\right)),$$

$$\partial_p D_T = \partial D \times (0,T] \cup D \times \{t=0\},$$

$$d_p(x,t;y,s) = \|x-y\| + |t-s|^{\frac{1}{2}} \text{ where } \|\| \text{ denotes Euclidean distance in } \mathbb{R}^n.$$

(Remark: It is not hard to show that any caloric measure whose associated operator has L^∞ coefficients satisfies a bottom doubling condition, i.e. $\exists\, C > 0$ independent of (Q_0,s_0) and r so that if $\Delta_{b,\frac{r}{2}}(Q_0,s_0) = \{(Q,s) \in \partial_p D_T ; \|Q-Q_0\| < \frac{r}{2}$ and $s_0 - r^2 < s < s_0\}$, then $\omega\left(\Delta_r\left(Q_0,s_0\right)\right) \leq C\omega\left(\Delta_{b,\frac{r}{2}}\left(Q_0,s_0\right)\right)$).

Using a standard estimate for the Green's function $G(x,t;y,s)$, Fabes, Garofalo and Salsa [8] showed that the center doubling condition for $d\omega(Q,s)$ is equivalent to the backwards Harnack inequality for $G(x,t;y,s)$ at the lateral boundary of D_T. These estimates allow them to compare the non-tangential maximal function of a solution with the Hardy-Littlewood maximal function and then use standard arguments to obtain a Fatou theorem [12].

An attempt to adapt B. Dahlberg's result for elliptic measures [4] to the parabolic setting, in other words to find conditions which will imply the absolute continuity of two caloric measures, led me to consider the possibility of using a Carleson-type condition on two measures to obtain non-tangential convergence for a solution instead of trying to prove a center-doubling condition. The stopping-time argument in the proof of Lemmas 2.9 and 2.10 in Fefferman, Kenig and Pipher's paper [9] is the basis for the proofs of the two theorems which are presented here. Fefferman, Kenig and Pipher proved B. Dahlberg's result on absolute continuity for elliptic measures by a different method than the one originally presented by B. Dahlberg and under weaker assumptions.

The first section of this paper obtains a Fatou theorem—Corollary 2—from the L^2 inequality of Theorem 1 by standard methods. Theorem 1 says that if two parabolic operators $\frac{\partial}{\partial t} - L_0$ and $\frac{\partial}{\partial t} - L_1$ have coefficients satisfying a Carleson-type condition (C_1) below) and the measure ω_0 associated to one operator, $\frac{\partial}{\partial t} - L_0$, is a center doubling measure, then an L^2 inequality for solutions to the other operator, $\frac{\partial}{\partial t} - L_1$, can be proved.

The measure ω_1 associated to $\frac{\partial}{\partial t} - L_1$ is not assumed to be a center doubling measure. The L^2 inequality $\|Nu_1\|_{L^2(d\omega_0,\partial_p D_T)} \leq C\|f\|_{L^2(d\omega_0,\partial_p D_T)}$ where $Nu_1(Q,s)$ is the non-tangential maximal function of $u_1(x,t)$ and $u_1\Big|_{\partial_p D_T} = f$, then gives absolute continuity of ω_1 with ω_0 (Corollary 1) and a Fatou theorem for $u_1(x,t)$. I am indebted to Thomas Wolff for pointing out the argument which gives the Fatou theorem from Theorem 1; it replaces a much longer argument I had originally used.

The main result of this paper is in section 2. Section 2 presents a second Fatou theorem and its proof. In Theorem 2, there is no doubling assumption on either caloric measure ω_0 or ω_1. Instead a Carleson-type condition with vanishing trace is assumed, (C_2) (see B. Dahlberg's original Carleson condition in [4]), and non-tangential convergence to boundary data is assumed for one operator's solution; given these conditions, the other operator's solution also converges non-tangentially.

1.

Definitions and Notation. $D_T = D \times [0,T]$ is a smooth cylinder in $\mathbb{R}^{n+1}$, say $D = B_1^n(0)$ the unit ball in $\mathbb{R}^n$, $\partial_p D_T = \partial D \times [0,T] \cup D \times \{0\}$. The operators

$$L_0 = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_j}\right), \quad L_1 = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(b_{ij}(x,t)\frac{\partial}{\partial x_j}\right)$$

are strictly elliptic divergence form operators whose coefficients are time dependent, bounded and measurable.

For $f \in L^2(d\omega_0, \partial_p D_T)$ we assume that

$$\left(\frac{\partial}{\partial t} - L_0\right) u_0 = 0 \text{ in } D_T, \quad u_0 \,|_{\partial_p D_T} = f,$$

$$\left(\frac{\partial}{\partial t} - L_1\right) u_1 = 0 \text{ in } D_T, \quad u_1 \,|_{\partial_p D_T} = f,$$

$$F(x,t) = u_1(x,t) - u_0(x,t)$$

(see [7] for the existence of such solutions given continuous boundary data).

$\omega_0 = \omega_0^{(X_0,T_0)}$ and $\omega_1 = \omega_1^{(X_0,T_0)}$ are the caloric measures for the operators $\frac{\partial}{\partial t} - L_0$ and $\frac{\partial}{\partial t} - L_1$, and $G_0(x,t;y,s)$, $G_1(x,t;y,s)$ are the Green's functions on D_T.

$$T_\rho(Q,s) = \{(y,t) \mid d_p(y,t;Q,s) < \rho\}; \Delta_\rho(Q,s) = T_\rho(Q,s) \cap \partial_p D_T,$$

$$\Delta_r^t(Q,s) = \Delta_r\left(Q, s + (2+\eta) r^2\right),$$

$$\Delta_r^b(Q,s) = \Delta_r\left(Q, s - (2+\eta) r^2\right),$$

$$d_p(y,t;x,s) = |x-y| + |t-s|^{1/2},$$

$|\ \ |$ = Euclidean metric in $\mathbb{R}^n$, $\mathbb{R}^1$,

$$\delta(x,t) = d_p(x,t;\partial_p D_T),$$

$$\Gamma_\alpha(Q,s) = \{(y,t) \in D_T : d_p(y,t;Q,s) \le \alpha\delta(y,t)\},$$

$$S_\beta(u)(Q,s) = \left(\int_{\Gamma_\beta(Q,s)} |\nabla_y u(y,t)|^2 \delta(y,t)^{-n}\, dydt\right),$$

$$M_{\omega_0}(f)(Q,s) = \sup_{\substack{\Delta_r \ni (Q,s) \\ r \le r_0}} \frac{1}{\omega_0(\Delta_r)} \int_{\Delta_r} \left|f\left(\widehat{Q},\widehat{s}\right)\right| d\omega_0\left(\widehat{Q},\widehat{s}\right)$$

is the usual (non-centered) Hardy-Littlewood maximal function with respect to caloric measure and

$$M^{t.l.}_{\omega_0}(f)(Q,s) = \sup_{\substack{r\le r_0\\ (Q_0,s_0)\in\Delta_r(Q,s)}} \frac{1}{\omega_0\left(\Delta_r\left(Q,s+ar^2\right)\right)}\int_{\Delta_r(Q,s)} f\left(\widehat{Q},\widehat{s}\right)d\omega_0\left(\widehat{Q},\widehat{s}\right),$$

$$[\varepsilon_{ij}(x,t)] = [a_{ij}(x,t) - b_{ij}(x,t)],$$

$$\varepsilon(x,t) = \sup_{i,j}|a_{ij}(x,t) - b_{ij}(x,t)|,$$

$$a(y,s) = \sup_{(x,t)\in P_{\frac{\delta(x,s)}{2}}(x,s)} |\varepsilon(x,t)|,$$

$$P_{\frac{\delta(x,t)}{2}}(x,t) = \left\{(y,s)\in D_T : d_p(y,s;x,t) < \frac{\delta(x,t)}{2}\right\},$$

$$N_\alpha(u)(Q,s) = \sup_{(x,t)\in\Gamma_\alpha(Q,s)}|u(x,t)|,$$

$$\widetilde{N}_\alpha(u)(Q,s) \quad = \quad \sup_{(x,t)\in\Gamma_\alpha(Q,s)}\left(\frac{1}{\left|P_{\frac{\delta(x,t)}{2}}(x,t)\right|}\int_{P_{\frac{\delta(x,t)}{2}}(x,t)}|u(y,t)|^2\,dyd\tau\right)^{1/2}.$$

Theorem 1. *If there is an* $\varepsilon_0 > 0$ *sufficiently small so that for all* $(Q,s)\in \partial p D_T$

(C$_1$) $$\sup_{\rho\le r_0}\left(\frac{1}{\omega_0^{(X_0,T_0)}\left(\Delta_\rho(Q,s)\right)}\int_{T_\rho(Q,s)\cap D_T}\frac{G_0(X_0,T_0;y,s)\,a^2(y,s)}{\delta^2(y,s)}dyds\right) < \varepsilon_0,$$

then whenever ω_0 *satisfies a center doubling condition there is a constant* c, $c = c(\lambda,n,r_0,\varepsilon_0,\alpha_0)$, *such that*

(D) $$\|N_{\alpha_0}u_1\|_{L^2(d\omega_0,\partial_pD_T)} \le c\,\|f\|_{L^2(d\omega_0,\partial_pD_T)}.$$

Corollary 1. ω_1 *is absolutely continuous with* ω_0 *on* ∂_pD_T.

Corollary 2. $\lim_{\substack{(x,t)\to(Q,s)\\(x,t)\in\Gamma_\alpha(Q,s)}} u_1(x,t) = f(Q,s)$ *a.e.* $d\omega_0$.

Corollary 2 to Theorem 1 is the doubling Fatou theorem.

Theorem 1 is an extension of a theorem proved in [13] where

(D) $$\|Nu_1\|_{L^2(d\omega_0,\partial_pD_T)} \le c\,\|f\|_{L^2(d\omega_0,\partial_pD_T)}$$

was shown to hold on the case where $L_0 \equiv L_1$ on $D\times\left[0,\delta_0^2\right]$ and the Carleson condition (C$_1$) was therefore only assumed to hold on $D\times\left[\delta_0^2,T\right]$.

The proof of Theorem 1 is very similar to the proof of the previous result. For the sake of clarity, an outline of the argument is included here.

Proof of Theorem 1. Theorem 1 is proved by adapting the argument in the proof of Theorem 2.5 in [9] to the parabolic setting. The following versions of Lemmas 2.9 and 2.10 in [9]

Lemma 1. $\widetilde{N}(F)(Q,s) \le C_1\epsilon_0 M_{\omega_0}(S(u_1))(Q,s)$ *and* $\|\widetilde{N}(\delta\nabla F)\|_{L^2(d\omega_0,\partial_pD_T)} \le C_3\epsilon_0\|S(u_1)\|_{L^2(d\omega_0,\partial_pD_T)}$.

Lemma 2.

$$\|S(F)\|_{L^2(d\omega_0,\partial_p D_T)} \le C_4(\|\widetilde{N}(F)\|_{L^2(d\omega_0,\partial_p D_T)} + \|\widetilde{N}(\delta\nabla F)\| + \|f\|_{L^2(d\omega_0,\partial_p D_T)}).$$

along with the inequalities

(1) $\|S(u_0)\|_{L^2(d\omega_0,\partial_p D_T)} \le C_2 \|f\|_{L^2(d\omega_0,\partial_p D_T)}$,
(2) $\|N(u_0)\|_{L^2(d\omega_0,\partial_p D_T)} \le C_5 \|f\|_{L^2(d\omega_0,\partial_p D_T)}$

yield the desired result (D), again by using the argument in [9], p. *78*.

Lemmas 1 and 2 are also obtained by adapting the proofs of Lemmas 2.9 and 2.10 in [9] to the parabolic setting. The details of the adaptations are long and technical. They appear in another paper where a version of Theorem 1 is proved on the lateral boundary of D_T in full [13], so they are not repeated here. The essential ingredients in adapting the arguments in [9] to the parabolic setting are using parabolic cubes in place of ordinary cubes, the energy estimate on parabolic solutions (p. 623 Lemma 1.1 [1]) in place of Cacciopoli's inequality for elliptic functions, Holder continuity for solutions which vanish on the lateral boundary $\partial_p^+ D_T = \partial D \times (0,T)$ (cf. [1], Theorems C & D), the comparison theorems for such solutions ([8], Theorems, 1.6 and 1.7) and standard estimates on the Green's function G_0 ([1], [8]). Assuming the Carleson-type condition on the full parabolic boundary $\partial_p D_T$ allows one to use the same stopping time argument (which is the key to the proofs of Lemmas 2.9 and 2.10 in [9]) across the bottom of $\partial_p D_T$ as well as on the lateral boundary. It is necessary to use the center-doubling property of ω_0 and backwards Harnack at the boundary for G_0 in several places to obtain Lemmas 1 and 2.

(1) is proved using Green's theorem and a standard argument on the area integral [5]. (2) follows from Theorem 2.13 in [8] and a standard argument involving the Hardy-Littlewood maximal function [12]. The center-doubling property of ω_0 is essential for obtaining both these inequalities.

Now Corollary 1 follows from inequality (D) taking $f(Q,s) = \chi_E(Q,s)$, and using estimates on the Radon-Nikodym derivative $\frac{d\omega_1^{(x,t)}}{d\omega_1^{(x_0,T_0)}}(Q,s)$. (See pages 4 & 5 of [13].) Corollary 2 also follows from (D) by a standard argument [12].

2.

For Theorem 2, the operators L_0, L_1 are assumed to be the same if $t \le \delta_0^2$.

Theorem 2. *If there is an* $\varepsilon(r)$ *so that for all* $r \le r_0$ *and* $(Q,s) \in \partial_p D_T \cap \{D \times [\delta_0^2, T]\}$

(C$_2$)

$$\sup_{\rho \le r}\left(\frac{1}{\omega_0\left(\Delta_\rho\left(Q; s+2_{\rho^2}\right)\right)}\int_{T_\rho(Q,s)\cap D_T}\left|\frac{G_0(X_0,T_0;y,s)\,a^2(y,s)}{\delta^2(y,s)}\right| dyds\right)^{1/2} \le \varepsilon(r)$$

and if $M_{\omega_0}^{t.l.}(S(u_1))(Q,s) < \infty$ *a.e.* $d\omega_0$*, then* $\lim_{\substack{(x,t)\to(Q,s)\\(x,t)\in\Gamma_\alpha(Q,s)}} u_1(x,t) = f(Q,s)$ *a.e.* $d\omega_0$ *whenever* $\lim_{\substack{(x,t)\to(Q,s)\\(x,t)\in\Gamma_\beta(Q,s)}} u_0(x,t) = f(Q,s)$*, for some* $\beta > \alpha$*.*

Here $\varepsilon(r) \le cr^\gamma$, $2 < \gamma < 2 + \frac{\alpha_0}{2}$ *where* $\alpha_0 =$ *Hölder constant for* L_0 *and* L_1*.*

Theorem 2 assumes no doubling condition on either caloric measure; it is necessary to assume the Fatou theorem holds for one operator's solution to obtain the same result for the other solution, given a Carleson condition which has a time lag and is of vanishing trace, i.e. (C_2) is considerably stronger than (C_1). At present I do not know how to obtain an inequality of the form (D) unless ω_0 is a center doubling measure, so the argument to prove Theorem 2 is necessary to obtain a Fatou theorem when no center doubling condition is assumed for either measure.

Proof of Theorem 2. The argument that follows uses the ideas of the proof of Lemma 2.9 in [9]. The method for obtaining the Riesz decomposition for parabolic functions in Doob [6] can be applied to elliptic-parabolic operators along with integration by parts and identities for solutions u_i to $\left(\frac{\partial}{\partial t} - L_i\right) u_i = 0$ to obtain the following integral representation of the difference between the two solutions:

$$u_1(x,t) - u_0(x,t) = F(x,t) = \int_0^T \int_D \nabla_y G_0(x,t;y,s) \cdot [\varepsilon_{ij}(y,s)] \nabla_y u_1(y,s)\, dyds.$$

Fixing $(Q_0, s_0) \in \partial_p D_T,\ s_0 \geq \delta_0^2$ and taking $(x,t) \in \Gamma_\alpha(x,t),\ F(x,t)$ can be further subdivided as

$$F(x,t) = \int_{P_{\frac{\delta(x,t)}{2}}(x,t)} \nabla_y G_0 \cdot \varepsilon \cdot \nabla_y u_1 + \int_{D_T \setminus P_{\frac{\delta(x,t)}{2}}(x,t)} \nabla_y G_0 \cdot \varepsilon \cdot \nabla_y u_1.$$

Now it is enough to show that $\widetilde{N}_h F(Q_0, s_0) \to 0$ as $h \to 0$ to obtain

$$\lim_{\Gamma_\alpha(Q_0,s_0) \ni (x,t) \to (Q_0,s_0)} N u_1(x,t) = f(Q_0, s_0)$$

whenever

$$\lim_{\Gamma_\beta(Q_0,s_0) \ni (x,t) \to (Q_0,s_0)} N u_0(x,t) = f(Q_0, s_0),$$

for some $\alpha < \beta$. To estimate $\widetilde{N}_h F(Q,s)$, one can estimate

$$\left(\frac{1}{\left| P_{\frac{\delta(x,t)}{2}}(x,t) \right|} \int_{P_{\frac{\delta(x,t)}{2}}} |F_1(y,s)|^2\, dyds \right)^{1/2}$$

and $F_2(x,t)$ for $(x,t) \in \Gamma(Q,s)$. Both these quantities will be shown to be $\leq \varepsilon(\delta(x,t))\ H(Q,s)$ where $\varepsilon(\delta(x,t)) \to 0$ as $(x,t) \to (Q,s)$ in $\Gamma(Q,s)$, and $H(Q,s) < \infty$ a.e. $d\omega_0$.

First

$$\left(\frac{1}{\left| P_{\frac{\delta(x,t)}{2}}(x,t) \right|} \int_{P_{\frac{\delta(x,t)}{2}}} |F_1(y,s)|^2\, dyds \right)^{1/2} \leq \varepsilon(\delta(x,t)) \cdot S u_1(Q,s)$$

by the argument in [13] and [9], since that argument did not depend on $G_0(x,t;y,s)$ satisfying a backwards Harnack condition. This means

$$\widetilde{N}_h F_1(Q,s) \leq \varepsilon(h)\, S u_1(Q,s). \tag{1}$$

To estimate $\widetilde{N}_h F_2(Q,s)$, $F_2(x,t)$ can be estimated pointwise. It is necessary to use a somewhat different approach from the one in [13] and [9] because the Carleson condition doesn't give the necessary decay in all parts of D_T.

First fix $(x,t) \in \Gamma(Q,s)$ and subdivide $D_T \backslash P_{\frac{\delta(x,t)}{2}}(x,t)$ into the regions:

$$\Omega_0 = \left\{ (y,s) \mid d_p(y,s;x^*,t^*) < \frac{\delta(x,t)}{2} \right\} \cap D_T$$

where (x^*,t^*) is the projection of (x,t) onto the lateral boundary of D_T (so $t = t^*$, x^* is the radial projection of x onto ∂D if D is the unit ball of $\mathbb{R}^n$),

$$\Omega_{j+1} = \left\{ (y,s) \mid 2^{(j-1)}\delta(x,t) \le d_p(y,s;x^*,t^*) < 2^j \delta(x,t) \right\} \cap D_T$$

for $j = 0,1,2,\cdots,N-1$ where N is chosen so that $2^N \delta(x,t) = c_0$, a fixed constant (say $c_0 = \frac{1}{8}$ for example),

$$R_{\delta(x,t)} = \left\{ (y,s) \mid d(y,\partial D) < \frac{\delta(x,t)}{2}; s \ge \delta_0^2 \right\} \cap \Omega_0^c \cap D_T,$$

$$D_j = \left\{ (y,s) \mid d(y,\partial D) \sim 2^j \delta(x,t), s \ge \delta_0^2 \right\} \cap \left(\bigcup_{j=0}^{n} \Omega_j \right)^c \cap D_T, j = 1,2,\cdots,N,$$

$$\Omega_{\frac{1}{8}} = D_T \backslash \left\{ \bigcup_{j=0}^{N-1} \Omega_j \cup R_{\delta(x,t)} \cup \bigcup_{j=1}^{N} D_j \right\},$$

so

$$\Omega_{\frac{1}{8}} = \left\{ (y,s) \in D_T \mid \delta(y,s) \ge c_0 \right\}.$$

Now write

$$\begin{aligned} F_2(x,t) &= \int_{D_T \backslash P_{\frac{\delta(x,t)}{2}}(x,t)} \nabla_y G_0 \cdot \varepsilon \cdot \nabla_y u_1 1 \\ &= \int_{\Omega_0} + \sum_{j=1}^{N} \int_{\Omega_j} + \int_{R_{\delta(x,t)}} + \sum_{j=1}^{n} \int_{D_j} + \int_{\Omega_{\frac{1}{8}}} \end{aligned}$$

and then each of these integrals can be bounded as follows:

The integrals over Ω_0, $R_{\delta(x,t)}$ and $\Omega_j \cap R_{\delta(x,t)}$ can be estimated using a stopping time argument similar to the one in Fefferman, Kenig and Pipher [9] as adapted to parabolic functions [13]. However, the fact that $G_0(x,t;y,s)$ no longer satisfies a boundary backwards Harnack condition (and a fact which is equivalent to this, namely that ω_0 does not satisfy a center doubling condition [8]) means a time lag must be introduced. This necessitates having a time lag in the Carleson condition (C_2) and also means the time lag maximal function must be used in place of the ordinary Hardy-Littlewood maximal function. Notice that (C_2)$\Rightarrow$(C_1) and $M_{\omega_0}^{t.l.}(f)(Q,s) \ge M_{\omega_0}(f)(Q,s)$. These expressions are equivalent if ω_0 is a center doubling measure. The stopping time argument for the region Ω_0 appears in Appendix A* at the end of this paper. It gives

$$\left| \int_{\Omega_0} \nabla_y G \cdot \varepsilon \cdot \nabla_y u_1 \right| \le c\varepsilon(\delta(x,t)) M_{\omega_0}^{t.l.}(Su_1(Q_0,s_0)). \tag{2}$$

In the regions Ω_j, Hölder continuity at the boundary of D_T is used on $G_0(x,t;y,s)$, $(y,s) \in \Omega_j$, so $G_0(x,t;y,s) \ne 0$ only for $t > s$. Pick $(x_j,t_j) \in \Omega_{j+1}$, say where (x_j, t_j) is the $\overline{A}_{r_j}(\cdot,\cdot)$ point for the projection of $\Omega_j \cap \{s \leq t\}$ onto $\partial_p D_T = \Delta_j$. Then

$$|G_0(x,t;y,s) - G_0(x^*,t^*;y,s)| \leq \max_{\Omega_j \cap \{s\leq t\}} |G(\cdot,\cdot;y,s)| \left(\frac{d_p(x,t;x^*,t^*)}{2^j \delta(x,t)}\right)^\alpha$$

$$= c\cdot 2^{-j\alpha} \max_{\Omega_j \cap \{s\leq t\}} |G(\cdot,\cdot;y,s)| = c2^{-j\alpha}|G_0(x_j,t_j;y,s)|$$

by the Carleson box estimate (cf. [8], Theorem 0.3) on G_0, for any point $(y,s) \in \Omega_j$.

For $\Omega_j \cap R_{\delta(x,t)}$ one can apply the stopping time argument (Appendix A*) using the estimate (obtained by using Hölder continuity and the global comparison principle (Theorem 1.7 in [8]))

$$G_0(x,t;y,s) \leq c\cdot 2^{-j\alpha} \frac{G_0(X_0,T_0;y,s)}{\omega_0^{X_0,T_0}\left(\Delta_j^t\right)}$$

in the appropriate place to find that

$$\left|\int_{\Omega_j \cap R_{\delta(x,t)}} \nabla G_0 \cdot \varepsilon \cdot \nabla u_1\right| \leq c2^{-j\alpha}\varepsilon(\delta(x,t))\, M^{t.l.}_{\omega_0}(S(u_1))(Q_0,s_0) \tag{3}$$

because $\delta(y,s) \sim \delta(x,t)$, for $(y,s) \in R_\delta(x,t)$.

For the remaining region at the boundary $R_{\delta(x,t)} \cap \left(\bigcup_{j=0}^N \Omega_j\right)^c$, a fixed set of boundary disks or cubes $\Delta_m(Q_i,s_i)$ can be chosen so that $\bigcup_{i=1}^{\ell(m)} \Delta_m(Q_i,s_i) = \partial_p^+ D_{2T}$, $m = m(\delta_0, T)$ is a uniform radius and the points Q_i, s_i are such that $M^{t.l.}_{\omega_0} Su_1(Q_i,s_i) < \infty$. Again using local comparison (Theorem 1.6 in [8]) on $G_0(x,t;y,s) \leq c\frac{G(X_0,T_0;y,s)}{\omega_0(\Delta_m^t(Q_i,s_i))}$ and the stopping time argument on each boundary region $T_m^i \subseteq R_{\delta(x,t)}$ where $T_m^i \cap \partial_p^+ D_T = \Delta_m^t(Q_i,s_i)$ gives

$$\left|\int_{R_{\delta(x,t)} \cap \left(\bigcup_{j=1}^N \Omega_j\right)^c} \nabla_y G_0(x,t;y,s)\, \varepsilon_{ij}(y,s)\, \nabla_y u_1(y,s)\, dyds\right| \tag{4}$$

$$\leq c\varepsilon(\delta(x,t)) \sum_{i=1}^{\ell(m)} M^{t.l}_{\omega_0} S(u_1)(Q_i,s_i).$$

(Q_i,s_i) are fixed, which means this expression $\to 0$ as $\delta(x,t) \to 0$.

The remaining regions lie away from $\partial_p^+ D_T$ so one must keep a factor of $|G_0(x,y;y_0,0)|$ in order to obtain an upper bound which $\to 0$ as $\delta(x,t) \to 0$.

In $\Omega_j \cap \Gamma(Q_0,s_0)$ let (x_j,t_j) be a fixed point at the upper right hand corner of $\Omega_j \cap \Gamma$, and (y_j,s_j) be the corresponding point at the lower right hand corner $\delta(x_j,t_j) \sim 2^{+j}\delta(x,t) = r_j \sim \delta(y_j,s_j)$. Then

$$\left|\int_{\Omega_j \cap \Gamma} \nabla G_0 \cdot \varepsilon \cdot \nabla u_1\right| \leq a\left(\frac{x_j+y_j}{2}, \frac{t_j+s_j}{2}\right)$$

$$\cdot \left(\int_{\Omega_j^* \cap \Gamma_\alpha} |G_0(x,t;y,s)|^2 \frac{dyds}{\delta_j^2}\right)^{1/2} \left(\int_{\Omega_j \cap \Gamma} |\nabla u_1|^2\right)^{1/2}$$

by Cauchy-Schwarz and the energy estimate on G_0. Using Hölder continuity on G_0 in the adjoint variable and Harnack in both the adjoint and forward variables the above is

$$\leq \left(2^{-(N-j)\alpha/2}\left(\left|G_0\left(x,t;y_0,0\right)\right|^{1/2}\right)\right)\left(\int_{\Omega_j^*\cap\Gamma}\frac{\left|G_0\left(x_j,t_j;y,s\right)\right|}{\delta^2\left(y,s\right)}\cdot a^2\left(y,s\right)dyds\right)^{1/2}$$

$$\left(\int_{\Omega_j\cap\Gamma}\left|\nabla u_1\right|^2\right)^{1/2}=A.$$

Again by the comparison principle (Theorem 1.6 in [8])

$$\left|G_0\left(x_j,t_j;y,s\right)\right|\leq\frac{c\cdot\left|G_0\left(X_0,T_0;y,s\right)\right|}{\omega_0^{X_0,T_0}\left(\Delta_j^t\right)}$$

so

$$\begin{aligned}A&\leq 2^{-(N-j)\alpha/2}\left|G_0\left(x,t;y_0,0\right)\right|^{1/2}\cdot\varepsilon\left(2^j\delta\left(x,t\right)\right)\cdot\left(\int_{\Omega_j\cap\Gamma}\left|\nabla u_1\right|^2\right)^{1/2}\\&\leq c\cdot 2^{-(N-j)\alpha/2}\cdot\varepsilon_0\cdot\left|G_0\left(x,t;y_0,0\right)\right|^{1/2}\cdot S\left(u_1\right)\left(Q_0,s_0\right),\end{aligned}$$

and summing over j gives

$$\left|\int_{\bigcup_{j=1}^{N}\Omega_j\cap\Gamma(Q_0,s_0)}\nabla_y G_0\left(x,t;y,s\right)\cdot\varepsilon_{ij}\left(y,s\right)\cdot\nabla_y u_1\left(y,s\right)dyds\right|$$

$$\leq c\cdot\left|G_0\left(x,t;y_0,0\right)\right|^{1/2}Su_1\left(Q_0,s_0\right).$$

For the regions D_j again subdivide $\bigcup_{j=1}^{N}D_j\cap\Gamma^c_{(Q_0,s_0)}$ into pieces of cylindrical annuli T_i^M which correspond to $\Delta_M\left(Q_i,s_i\right)$. By adding Q_0, s_0 to $\bigcup_{i=1}^{\ell(m)}\left(Q_i,s_i\right)$ one can handle $\Omega_j\cap\Gamma\left(Q_0,s_0\right)^c$ here as well. Now

$$\begin{aligned}\left|\int_{T_i^M\cap D_j}\nabla_y G_0\cdot\varepsilon\cdot\nabla_y u_1\right|&\leq c2^{-j\alpha}\sum_k\left(\int_{T_j^{k*}}\frac{\left|G_0\left(X_0,T_0;y,s\right)\right|}{\delta^2\left(y,s\right)}\cdot a^2\left(y,s\right)dyds\right)^{1/2}\\&\cdot\frac{1}{\omega_0^{(X_0,T_0)}\left(\Delta_{j,k}^t\right)}\left|G_0\left(X_0,T_0;y_{j,k},s_{j,k}\right)\right|^{1/2}\left(\int_{T_j^k}\left|\nabla u_1\left(y,s\right)\right|dyds\right)^{1/2}\\&\leq\frac{c\cdot 2^{-j\alpha}\varepsilon\left(\delta_j\right)}{\omega_0^{X_0,T_0}\left(\Delta_M^t\left(Q_i,s_i\right)\right)}\sum_\ell\left(\sum_{\tau_\ell}\omega_0\left(I_j^{k,b}\right)\right)^{1/2}\\&\cdot\left(\sum_{\tau_\ell}\frac{\omega_0\left(I_j^{k,b}\right)}{\ell\left(I_j^k\right)^n}\int_{T_j^k}\left|\nabla u_1\left(y,s\right)\right|^2dyds\right)^{1/2};\end{aligned}$$

again using the stopping time argument decomposition (see Appendix A*) the above is

$$\leq \frac{c \cdot 2^{-j\alpha} \varepsilon\left(\delta_j\right)}{\omega_0\left(\Delta_M^t\right)} \cdot \int_{\Delta_M} S_\beta u_1\left(Q, s\right) d\omega_0^{X_0, T_0}\left(Q, s\right)$$

$$\leq c \cdot 2^{-j\alpha} 2^{j\gamma} \delta\left(x, t\right)^\gamma M_{\omega_0}^{t.l.} S_\beta u_1\left(Q_i, s_i\right),$$

so if $\gamma < \alpha$ these integrals sum over j to give

$$\left|\sum_{j=1}^{N} \int_{D_j \cap \Gamma(Q,s)^c} \nabla_y G_0 \cdot \varepsilon_{ij} \cdot \nabla_y u_1\right| \leq c\delta\left(x, t\right)^\gamma \cdot \sum_{j=0}^{\ell(m)} M_{\omega_0}^{t.l.} S_\beta u_1\left(Q_j, s_j\right). \tag{5}$$

For the last region $\Omega_{\frac{1}{8}}$ an argument similar to the one for $\Omega_j \cap \Gamma\left(Q, s\right)$ yields an upper bound of

$$\left|\int_{\Omega_{\frac{1}{8}}} \nabla_y G_0 \cdot \varepsilon \cdot \nabla_y U_1\right| \leq c\left|G_0\left(x, t; y_0, 0\right)\right|^{1/2} \sum_{j=0}^{\ell(m)} S u_1\left(Q_j, s_j\right). \tag{6}$$

Putting (1)-(6) together it is easy to see that $F_2\left(x, t\right) \to 0$ as $\left(x, t\right) \to \left(Q_0, s_0\right)$ non-tangentially so that $\widetilde{N}_{\delta(x,t)} F_2\left(Q_0, s_0\right) \to 0$ as $\delta\left(x, t\right) \to 0$.

Appendix A*. Stopping time argument for a non-doubling measure.

$$\left(Q_0, s_0\right) \in \partial_p^+ D_T \times \left[\delta_0^2, T\right]; r \leq \frac{r_0}{2};$$

$$2\Delta_r\left(Q_0, s_0\right) \subseteq \partial_p^+ D_T, \delta\left(x, t\right) = r \text{ for } \left(x, t\right) \in \Gamma_\alpha\left(Q_0, s_0\right).$$

$\left(x^*, t^*\right)$ = projection of $\left(x, t\right)$ onto $\partial_p^+ D_T$ so that $t^* = t$.

$$\Omega_0 = \left\{\left(y, s\right) \mid d_p\left(y, s; x^*, t^*\right) < \frac{\delta\left(x, t\right)}{2}\right\} \cap D_T,$$

$$\Delta_0\left(x^*, t^*\right) = \Delta_0 = \Omega_0 \cap \partial_p^+ D_T.$$

Ω_0 can be covered by parabolic boxes whose dimension compares with their distance from $\partial_p^+ D_T \cup D \times \left\{t = t^* - \delta^2(x,t)/4\right\}$ so that the projection of each box onto $\partial_p^+ D_T$ is a dyadic surface cube which lies inside $\overline{\Delta}_0$. Let $\bigcup I_j^k =$ the union of all such dyadic surface cubes and $\bigcup T_j^k =$ the union of the corresponding parabolic boxes in D_T. Notice $\Omega_0 \subseteq \bigcup T_j^k$ and $\Delta_0 = \bigcup I_j^k$.

Now write

$$\left|\int_{\Omega_0} \nabla_y G_0(X_0, T_0; y, s) \cdot \varepsilon_{ij}(y,s) \cdot \nabla_y u_1(y,s)\, dyds\right|$$

$$\leq \sum_{\substack{j,k \\ T_j^k \cap \Omega_0 \neq \emptyset}} \left|\int_{T_j^k} \nabla G_0 \cdot \varepsilon \cdot \nabla u_1\right|$$

$$\leq c \sum_{j,k} \left(\int_{(1+\eta)T_j^k} \frac{|G_0(x,t;y,s)|^2}{\delta(y,s)^2} dyds\right)^{1/2} \sup_{(y,s)\in T_{j,k}} |\varepsilon(y,s)|$$

$$\cdot \left(\int_{T_j^k} |\nabla u_1(y,s)|^2\, dyds\right)^{1/2} = A$$

by Cauchy-Schwartz and the energy estimate. Now if $(y,s) \in \Omega_0$

$$\left|\frac{G_0(x,t;y,s)}{G_0(X_0,T_0;y,s)}\right| \leq c \left|\frac{G_0(x,t;\underline{A}_r(x^*,t^*))}{G_0\left(X_0,T_0;\overline{A}_r(x^*,t^*)\right)}\right|$$

by the local comparison theorem applied to the adjoint variable of G_0 (Theorem 1.6 in [8]), valid for all strictly elliptic L_0. By Theorem 1.5 in [8]

$$\left|\frac{G_0(x,t;\underline{A}_r(x^*,t^*))}{G_0\left(X_0,T_0;\overline{A}_r(x^*,t^*)\right)}\right| \leq c' \frac{\delta(x,t)^n\, \omega^{(x,t)}\left(\Delta_r\left(x^*, t^* - \alpha r^2\right)\right)}{\delta(x,t)^n\, \omega^{X_0,T_0}\left(\Delta_r(x^*, t^* + \beta r^2)\right)}$$

$$\leq \frac{c'}{\omega^{X_0,T_0}\left(\Delta_r(x^*, t^* + \beta r^2)\right)}.$$

c' depends on β and the constant in the definition of $\overline{A}_r$ $\underline{A}_r$ points.

So

$$|G_0(x,t;y,s)| \leq c \frac{|G_0(X_0,T_0;y,s)|}{\omega_0^{X_0 T_0}\left(\Delta_0^t(x^*,t^*)\right)}.$$

Remark. $\Delta_0^t \cap \Delta_0 = \emptyset$ and in fact Δ_0^t must lie slightly above Δ_0; the constants in this section depend on η (see Definitions and Notation).

Also by Theorem 1.5 in [8]

$$(y,s) \in (1+\eta)\, T_j^k \Rightarrow G_0(X_0, T_0, y, s) \leq \frac{c\omega_0^{(X_0,T_0)}\left(\Delta_j^{k,b}\right)}{\ell\left(I_j^k\right)^n},$$

so as in [9] one obtains

$$A \leq \frac{c}{\omega^{X_0,T_0}\left(\Delta_0^t\left(x^*,t^*\right)\right)} \sum_{\ell} \sum_{\tau_\ell} \left(\int_{(1+\eta)T_j^k} \frac{G_0\left(X_0,T_0;y,s\right)a^2\left(y,s\right)}{\delta^2\left(y,s\right)} \right)^{1/2}$$

$$\cdot \left(\frac{\omega_0^{X_0,T_0}\left(\Delta_j^{k,b}\right)}{\ell\left(I_j^k\right)^n} \int_{T_j^k} \left|\nabla U_1\left(y,s\right)\right|^2 dyds \right)^{1/2}$$

by writing the $\sum\limits_{j,k}$ as $\sum\limits_{\ell}\sum\limits_{\tau_\ell}$ where $I_j^k \in \tau_\ell$ and τ_ℓ is defined as follows:

Let

$$\mathcal{O}_\ell = \left\{ (Q,s) \in \Delta_0 : S_\beta\left(u_1\right)(Q,s) > 2^\ell \right\},$$

$$\widetilde{\mathcal{O}}_\ell = \left\{ (Q,s) \in \Delta_0 : M_{\omega_0}^d\left(\chi_{\mathcal{O}_\ell}\right)(Q,s) > \frac{1}{2} \right\}.$$

Then $\tau_\ell = \bigcup I_j^k$ such that

$$\omega_0^{X_0,T_0}\left(I_j^{k,b} \cap \mathcal{O}_\ell\right) > \frac{1}{2}\omega_0^{X_0,T_0}\left(I_j^{k,b}\right)$$

but $\omega_0\left(I_j^{k,b} \cap \mathcal{O}_{\ell+1}\right) \leq \frac{1}{2}\omega_0\left(I_j^{k,b}\right)$.

If $I_j^k \in \tau_\ell$, then $I_j^{k,b} \subseteq \widetilde{\mathcal{O}}_\ell$. Also

i) $I_j^k \in \tau_\ell \Rightarrow \omega_0\left(I_j^{k,b}\right) \leq 2\omega_0\left(I_j^{k,b} \cap \widetilde{\mathcal{O}}_\ell \backslash \mathcal{O}_{\ell+1}\right)$,

ii) $\omega_0\left(\widetilde{\mathcal{O}}_\ell\right) \leq 2\omega_0\left(\mathcal{O}_\ell\right)$;

both hold without ω_0 being a doubling measure (see [9]).

Using the Carleson condition on A, then

$$A \leq \frac{c}{\omega_0^{(X_0,T_0)}\left(\Delta_0^t\left(x^*,t^*\right)\right)} \sum_{\ell} \left(\sum_{\tau_\ell} \varepsilon^2\left(\delta\left(x,t\right)\right)\omega_0^{X_0,T_0}\left(\Delta_j^{k,b}\right) \right)^{1/2}$$

$$\cdot \left(\sum_{\tau_\ell} \omega_0^{(X_0,T_0)}\left(\Delta_j^{k,b} \cap \widetilde{\mathcal{O}}_\ell \backslash \mathcal{O}_{\ell+1}\right) \right) \left(\int_{T_j^k} \left|\nabla u_1\left(y,s\right)\right|^2 \delta\left(y,s\right)^{-n} dyds \right)^{1/2}$$

$$\leq \frac{c\varepsilon\left(\delta\left(x,t\right)\right)}{\omega_0\left(\Delta_0^t\right)} \sum_{\ell} \omega_0\left(\widetilde{\mathcal{O}}_\ell\right)^{1/2} \left(\int_{\widetilde{\mathcal{O}}_\ell \backslash \mathcal{O}_{\ell+1}} S_\beta\left(u_1\right)(Q,s)\, d\omega_0\left(Q,s\right) \right)^{1/2}$$

$$\leq \frac{c\varepsilon\left(\delta\left(x,t\right)\right)}{\omega_0\left(\Delta_0^t\right)} \sum_{\ell} 2^\ell \omega_0\left(\mathcal{O}_\ell\right) \leq \frac{c\varepsilon\left(\delta\left(x,t\right)\right)}{\omega_0\left(\Delta_0^t\right)} \int_{\Delta_0} S\left(u_1\right)(Q,s)\, d\omega_0\left(Q,s\right)$$

$$\leq c\varepsilon\left(\delta\left(x,t\right)\right) \cdot M_{\omega_0}^{t.l} S_\beta\left(u_1\right)\left(Q_0,s_0\right).$$

3.

Discussion: The Carleson-type condition in [9] adapts to the parabolic setting without too much difficulty. However to obtain inequality (D) of Theorem 1 it is essential that the measure ω_0 satisfy the center doubling condition. It is conjectured that the Carleson-type condition on L_0 and L_1, and the assumption that ω_0 satisfies a center-doubling condition, should be sufficient to obtain a center-doubling condition for ω_1. My inability to prove this conjecture led me to consider other ways of proving a Fatou theorem. If center doubling is proved for a general caloric measure, Theorem 2 will be obsolete; however, the method of obtaining the non-tangential convergence of $u_1(x,t)$ directly from the Carleson condition may prove useful in other settings where doubling is not known.

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