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ON COMPACT CONNECTED SETS IN BANACH SPACES

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ABSTRACT. Let $\mathbf{E}$ be a separable strictly convex Banach space of dimension at least 2. It is shown that there exists a nonempty compact connected set $X \subset \mathbf{E}$ such that the nearest point mapping $p_X : \mathbf{E} \to 2^{\mathbf{E}}$ is not single valued on a set of points dense in $\mathbf{E}$. Furthermore, it is proved that most (in the sense of the Baire category) nonempty compact connected sets $X \subset \mathbf{E}$ have the above property. Similar results hold for the furthest point mapping.

1. Introduction

Throughout this note $\mathbf{E}$ denotes a strictly convex Banach space with $\dim \mathbf{E} \geq 2$, and $2^{\mathbf{E}}$ the family of all nonempty subsets of $\mathbf{E}$. Additional assumptions about $\mathbf{E}$ will be made when needed. Set

$$\mathcal{C}(\mathbf{E}) = \{X \in 2^{\mathbf{E}} | X \text{ is compact and connected}\}.$$

The space $\mathcal{C}(\mathbf{E})$ is endowed with the Hausdorff distance h, under which it is a complete metric space.

For X a nonempty compact subset of $\mathbf{E}$, we consider the *nearest point mapping* $p_X : \mathbf{E} \to 2^{\mathbf{E}}$ and the *furthest point mapping* $q_X : \mathbf{E} \to 2^{\mathbf{E}}$ defined, respectively, by

$$p_X(a) = \{x \in X | \, \|x - a\| = d(a, X)\} \quad \text{and} \quad q_X(a) = \{x \in X | \, \|x - a\| = e(a, X)\},$$

where $d(a, X) = \inf\{\|x - a\| \, | x \in X\}$ and $e(a, X) = \sup\{\|x - a\| \, | x \in X\}$.

In a metric space M, an open (resp. closed) ball with center $x \in M$ and radius $r > 0$ is denoted by $B_M(x, r)$ (resp. $\widetilde{B}_M(x, r)$).

By a result of Stečkin [2], it is known that for each nonempty compact set $X \subset \mathbf{E}$, p_X is single valued on a residual subset of $\mathbf{E}$. On the other hand, Zamfirescu [3] has recently proved that for most (in the sense of the Baire category) nonempty compact sets $X \subset \mathbf{R}^d$, $d \geq 2$, the nearest point mapping p_X is not single valued on a set of points dense in $\mathbf{R}^d$. In this note it is shown that the result of Zamfirescu remains valid in any strictly convex separable Banach space $\mathbf{E}$. Furthermore, it is proved that for most nonempty compact connected sets $X \subset \mathbf{E}$, with $\mathbf{E}$ as before, the nearest point mapping is not single valued on a set of points dense in $\mathbf{E}$. Similar results hold for the furthest point mapping.

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2. Main results

Theorem 1. *Let* $\mathbf{E}$ *be a separable strictly convex Banach space, with* $\dim \mathbf{E} \geq 2$. *Let* $\mathcal{C}_0$ *be the set of all* $X \in \mathcal{C}(\mathbf{E})$ *such that the nearest point mapping* $p_X : \mathbf{E} \to 2^{\mathbf{E}}$ *is not single valued on a set of points dense in* $\mathbf{E}$. *Then* $\mathcal{C}_0$ *is a residual subset of* $\mathcal{C}(\mathbf{E})$.

Proof. Let $A \subset \mathbf{E}$ be countable and dense in $\mathbf{E}$. For $k \in \mathbf{N}$, let Q_k be the set of all rational numbers r with $0 < r < 1/(2k)$. For $a \in A$, $k \in \mathbf{N}$ and $r \in Q_k$, put:

$$\mathcal{N}_a = \{X \in \mathcal{C}(\mathbf{E}) | a \in X\},$$
$$\mathcal{N}_{a,k} = \{X \in \mathcal{C}(\mathbf{E}) | d(a, X) > 1/k\},$$
$$\mathcal{N}_{a,k,r} = \{X \in \mathcal{N}_{a,k} | p_X \text{ is single valued at each point of } B_{\mathbf{E}}(a, r)\}.$$

We claim that $\mathcal{N}_a$ is nowhere dense in $\mathcal{C}(\mathbf{E})$. Suppose the contrary. Since $\mathcal{N}_a$ is closed in $\mathcal{C}(\mathbf{E})$, there exist $X \in \mathcal{N}_a$ and $\varepsilon > 0$ such that $B_{\mathcal{C}(\mathbf{E})}(X, \varepsilon) \subset \mathcal{N}_a$. If a is an interior point of X we fix $0 < \tilde{\varepsilon} < \varepsilon$ so that $B_{\mathbf{E}}(a, \tilde{\varepsilon}) \subset X$ and define $Y = X \setminus B_{\mathbf{E}}(a, \tilde{\varepsilon}/2)$. Since $Y \in \mathcal{C}(\mathbf{E}) \setminus \mathcal{N}_a$ and $h(Y, X) < \varepsilon$, a contradiction follows. If a is in the boundary of X we take $u \in \mathbf{E} \setminus X$ such that $0 < \|u - a\| < \varepsilon/2$ and define $Y = X + a - u$. Since $Y \in \mathcal{C}(\mathbf{E}) \setminus \mathcal{N}_a$ and $h(Y, X) < \varepsilon$, again a contradiction follows. Hence $\mathcal{N}_a$ is nowhere dense in $\mathcal{C}(\mathbf{E})$.

Claim. $\mathcal{N}_{a,k,r}$ is nowhere dense in $\mathcal{C}(\mathbf{E})$.

Postponing the proof of this claim, define

$$\mathcal{N} = \left(\bigcup_{a \in A} \mathcal{N}_a \right) \cup \left(\bigcup_{a \in A} \bigcup_{k \in \mathbf{N}} \bigcup_{r \in Q_k} \mathcal{N}_{a,k,r} \right).$$

Clearly $\mathcal{N}$ is of the Baire first category in $\mathcal{C}(\mathbf{E})$. To complete the proof of the theorem it suffices to show that $\mathcal{C}(\mathbf{E}) \setminus \mathcal{N} \subset \mathcal{C}_0$.

Indeed, let $X \in \mathcal{C}(\mathbf{E}) \setminus \mathcal{N}$. Let $x \in \mathbf{E}$ and $\varepsilon > 0$ be arbitrary. Let $a \in A \cap B_{\mathbf{E}}(x, \varepsilon)$. Let $k \in \mathbf{N}$ be such that $d(a, X) > 1/k$, thus $X \in \mathcal{N}_{a,k}$.

Take $r \in Q_k$ satisfying $0 < r < \varepsilon - \|a - x\|$. Since $X \in \mathcal{N}_{a,k} \setminus \mathcal{N}_{a,k,r}$, there exists a point $u \in B_{\mathbf{E}}(a, r)$ such that $p_X(u)$ contains at least two different points. As $u \in B_{\mathbf{E}}(x, \varepsilon)$, and $x \in \mathbf{E}$ and $\varepsilon > 0$ are arbitrary, it follows that $X \in \mathcal{C}_0$. Hence $\mathcal{C}(\mathbf{E}) \setminus \mathcal{N} \subset \mathcal{C}_0$ and $\mathcal{C}_0$ is residual in $\mathcal{C}(\mathbf{E})$, for $\mathcal{C}(\mathbf{E}) \setminus \mathcal{N}$ is so. This completes the proof.

Theorem 2. *Let* $\mathbf{E}$ *be as in Theorem* 1. *Let* $\mathcal{C}_1$ *be the set of all* $X \in \mathcal{C}(\mathbf{E})$ *such that the furthest point mapping* $q_X : \mathbf{E} \to 2^{\mathbf{E}}$ *is not single valued on a set of points dense in* $\mathbf{E}$. *Then* $\mathcal{C}_1$ *is a residual subset of* $\mathcal{C}(\mathbf{E})$.

Proof. Let $A \subset \mathbf{E}$ be countable and dense in $\mathbf{E}$. Let Q^+ be the set of all strictly positive rationals. For $a \in A$ and $r \in Q^+$, define

$$\mathcal{M}_{a,r} = \{X \in \mathcal{C}(\mathbf{E}) | X \setminus \{a\} \neq 0 \text{ and } q_X \text{ is single valued at each point of } B_{\mathbf{E}}(a, r)\}.$$

Claim. $\mathcal{M}_{a,r}$ is nowhere dense in $\mathcal{C}(\mathbf{E})$.

Postponing the proof of this claim, define

$$\mathcal{M} = \left(\bigcup_{a \in A} \{a\} \right) \cup \left(\bigcup_{a \in A} \bigcup_{r \in Q^+} \mathcal{M}_{a,r} \right).$$

From the claim and the fact that each set $\{a\}$, with $a \in A$, is nowhere dense in $\mathcal{C}(\mathbf{E})$, it follows that $\mathcal{C}(\mathbf{E}) \setminus \mathcal{M}$ is residual in $\mathcal{C}(\mathbf{E})$. Furthermore, as in the proof of Theorem 1, one can show that $\mathcal{C}(\mathbf{E}) \setminus \mathcal{M} \subset \mathcal{C}_1$. Hence also $\mathcal{C}_1$ is residual in $\mathcal{C}(\mathbf{E})$, completing the proof. □

Set $\mathcal{K}(\mathbf{E}) = \{X \in 2^{\mathbf{E}} | X \text{ is compact}\}$. The space $\mathcal{K}(\mathbf{E})$ is endowed with the Hausdorff distance, under which it is a complete metric space.

The following Theorem 3 and Theorem 4 can be proved as Theorem 1 and Theorem 2, respectively. Theorem 3 generalizes a result obtained by Zamfirescu [3].

Theorem 3. *Let* $\mathbf{E}$ *be as in Theorem* 1*. Let* $\mathcal{K}_0$ *be the set of all* $X \in \mathcal{K}(\mathbf{E})$ *such that the nearest point mapping* $p_X : \mathbf{E} \to 2^{\mathbf{E}}$ *is not single valued on a set of points dense in* $\mathbf{E}$*. Then* $\mathcal{K}_0$ *is a residual subset of* $\mathcal{K}(\mathbf{E})$*.*

Theorem 4. *Let* $\mathbf{E}$ *be as in Theorem* 1*. Let* $\mathcal{K}_1$ *be the set of all* $X \in \mathcal{K}(\mathbf{E})$ *such that the furthest point mapping* $q_X : \mathbf{E} \to 2^{\mathbf{E}}$ *is not single valued on a set of points dense in* $\mathbf{E}$*. Then* $\mathcal{K}_1$ *is a residual subset of* $\mathcal{K}(\mathbf{E})$*.*

In view of Theorem 1 one has that most sets $X \in \mathcal{C}(\mathbf{E})$ are perfect and nowhere dense in $\mathbf{E}$. Moreover, from Theorem 3 and the fact that $\mathcal{C}(\mathbf{E})$ is nowhere dense in $\mathcal{K}(\mathbf{E})$, it follows that most sets $X \in \mathcal{K}(\mathbf{E})$ are perfect, not connected and nowhere dense in $\mathbf{E}$.

3. Completion of the proofs

The following two lemmas are used in order to complete the proof of Theorem 1.

Lemma 1. *Let* Z *be a normed space. Let* $a, u, v \in Z$ *be such that* $\|u - a\| = \|v - a\| = r$, $u - a \neq -(v - a)$*. Then, for every* $\lambda \in [0, 1]$,

$$\left\| \frac{\lambda(u-a) + (1-\lambda)(v-a)}{\|\lambda(u-a) + (1-\lambda)(v-a)\|} r + a - u \right\| \leq \|v - u\|.$$

Proof. It follows from Schäffer [1, p. 6].

Lemma 2. *Let* Z *be a normed space. Let* $a, u \in Z$, $a \neq u$*. For* $t \in \mathbf{R}$*, set* $v_t = a + t(u - a)$*. Then for every* $x \in B_{\mathbf{E}}(a, r/2)$*, where* $r = \|u - a\|$*, and for every* $t' > t \geq 1$ *we have*

$$\|v_t - x\| \leq \|v_{t'} - x\|.$$

Proof. In the contrary case there exist $x_0 \in B_{\mathbf{E}}(a, r/2)$ and $t_1 > t_0 \geq 1$ such that $\|v_{t_1} - x_0\| < \|v_{t_0} - x_0\|$. Clearly, $\|v_{t_0} - x_0\| \geq \|v_{t_0} - a\| - \|x_0 - a\| > r - r/2 > \|x_0 - a\|$. Let $0 < s < 1$ be such that $v_{t_0} = s v_{t_1} + (1 - s)a$. Then

$$\begin{aligned} \|v_{t_0} - x_0\| &= \|s v_{t_1} + (1-s)a - x_0\| \\ &\leq s\|v_{t_1} - x_0\| + (1-s)\|a - x_0\| \\ &< s\|v_{t_0} - x_0\| + (1-s)\|v_{t_0} - x_0\| = \|v_{t_0} - x_0\|, \end{aligned}$$

a contradiction. This completes the proof.

To complete the proof of Theorem 1 it remains to show that the following claim is true.

Claim. $\mathcal{N}_{a,k,r}$ is nowhere dense in $\mathcal{C}(\mathbf{E})$.

Proof. Indeed, let $X \in \mathcal{N}_{a,k,r}$ and $0 < \varepsilon < \min\{r, \mu - 1/k\}$ be arbitrary, where $\mu = d(a, X)$. Note that $\varepsilon < \mu/2$, for $\mu > 1/k > 2r > 2\varepsilon$. Using an idea from Zamfirescu [3], take $\tilde{y}_1 \in X$ and $\tilde{y}_2 \in \mathbf{E}$ satisfying $\|\tilde{y}_i - a\| = \mu$, $i = 1, 2$, and $\|\tilde{y}_1 - \tilde{y}_2\| = \varepsilon/8$. Observe that $\tilde{y}_2 - a \neq -(\tilde{y}_1 - a)$ since, in the contrary case, $\varepsilon/8 = \|(\tilde{y}_1 - a) - (\tilde{y}_2 - a)\| = 2\mu > 4\varepsilon$. Denote by $\gamma(\tilde{y}_1, \tilde{y}_2)$ the arc, with end points $\tilde{y}_1$ and $\tilde{y}_2$, given by

$$\gamma(\tilde{y}_1, \tilde{y}_2) = \left\{ \frac{(1-\lambda)(\tilde{y}_1 - a) + \lambda(\tilde{y}_2 - a)}{\|(1-\lambda)(\tilde{y}_1 - a) + \lambda(\tilde{y}_2 - a)\|}\mu + a \,|\, \lambda \in [0, 1] \right\}.$$

This is well defined, for $\tilde{y}_2 - a \neq -(\tilde{y}_1 - a)$. Furthermore, for $i = 1, 2$, set

$$y_i = \tilde{y}_i - \frac{\varepsilon}{4\mu}(\tilde{y}_i - a), \qquad u_i = a + \frac{\varepsilon}{8\mu}(\tilde{y}_i - a),$$

and define $Y = X \cup y_1\tilde{y}_1 \cup y_2\tilde{y}_2 \cup \gamma(\tilde{y}_1, \tilde{y}_2)$. Here $y_i\tilde{y}_i$, $i = 1, 2$, denotes the segment in $\mathbf{E}$ with end points y_i and $\tilde{y}_i$. Clearly $Y \in \mathcal{N}_{a,k}$, because $Y \in \mathcal{C}(\mathbf{E})$ and $d(a, Y) = \mu - \varepsilon/4 > 1/k$. If $y \in y_1\tilde{y}_1 \cup y_2\tilde{y}_2$ then $d(y, X) \leq 3\varepsilon/8$, while if $y \in \gamma(\tilde{y}_1, \tilde{y}_2)$, by Lemma 1 one has $d(y, X) \leq \|\tilde{y}_1 - \tilde{y}_2\| = \varepsilon/8$. Hence $h(Y, X) \leq 3\varepsilon/8 < \varepsilon/2$.

Set

(3.1)
$$\rho = \min\left\{ \frac{\|y_2 - u_1\| - \|y_1 - u_1\|}{2}, \frac{\|y_1 - u_2\| - \|y_2 - u_2\|}{2}, \frac{1}{3}\delta(y_1\tilde{y}_1, y_2\tilde{y}_2), \frac{\varepsilon}{16} \right\},$$

where $\delta(y_1\tilde{y}_1, y_2\tilde{y}_2) = \inf\{\|x_1 - x_2\| \,|x_1 \in y_1\tilde{y}_1,\, x_2 \in y_2\tilde{y}_2\}$. Observe that $\rho > 0$, because $\mathbf{E}$ is strictly convex and $\delta(y_1\tilde{y}_1, y_2\tilde{y}_2) > 0$. Furthermore, $B_{\mathcal{C}(\mathbf{E})}(Y, \rho) \subset \mathcal{N}_{a,k}$, since each $Z \in B_{\mathcal{C}(\mathbf{E})}(Y, \rho)$ satisfies $d(a, Z) \geq d(a, Y) - h(Z, Y) > \mu - \varepsilon/4 - \rho > \mu - \varepsilon > 1/k$.

Let $Z \in B_{\mathcal{C}(\mathbf{E})}(Y, \rho)$. Set

$$\widetilde{Z} = Z \cap \widetilde{B}_{\mathbf{E}}(a, \|y_1 - a\| + \rho), \qquad \widetilde{Z}_i = \widetilde{Z} \cap (y_i\tilde{y}_i + \rho\widetilde{B}), \quad i = 1, 2,$$

where $\widetilde{B} = \widetilde{B}_{\mathbf{E}}(0, 1)$. We have $\widetilde{Z}_i \neq \varnothing$, $i = 1, 2$, $\widetilde{Z}_1 \cap \widetilde{Z}_2 = \varnothing$, $\widetilde{Z}_1 \cup \widetilde{Z}_2 = \widetilde{Z}$.

Indeed, for $i = 1, 2$, $y_i \in Y \subset Z + \rho\widetilde{B}$; thus, for some $z_i \in Z$, $\|z_i - y_i\| \leq \rho$, which implies $\|z_i - a\| \leq \|y_i - a\| + \rho$. As $z_i \in \widetilde{Z}_i$, $i = 1, 2$, the sets $\widetilde{Z}_1$ and $\widetilde{Z}_2$ are nonempty. Let us show that $\widetilde{Z}_1 \cap \widetilde{Z}_2 = \varnothing$. Supposing otherwise, let $z \in \widetilde{Z}_1 \cap \widetilde{Z}_2$. Then, for some $v_i \in y_i\tilde{y}_i$ we have $\|v_i - z\| \leq \rho$, $i = 1, 2$, and so $\delta(y_1\tilde{y}_1, y_2\tilde{y}_2) \leq \|v_1 - v_2\| \leq \|v_1 - z\| + \|v_2 - z\| \leq 2\rho$, a contradiction. Hence $\widetilde{Z}_1 \cap \widetilde{Z}_2 = \varnothing$. It remains to prove that $\widetilde{Z}_1 \cup \widetilde{Z}_2 = \widetilde{Z}$. To this end, let $z \in \widetilde{Z}$. Then $z \in Z$ and $\|z - a\| \leq \|y_1 - a\| + \rho \leq \mu - \varepsilon/4 + \varepsilon/16 = \mu - 3\varepsilon/16$. Since $z \in Z \subset Y + \rho\widetilde{B}$, for some $y \in Y$ we have $\|z - y\| \leq \rho$. If $y \in X \cup \gamma(\tilde{y}_1, \tilde{y}_2)$, then $\|y - a\| \geq \mu$ and so

$$\rho \geq \|z - y\| \geq \|y - a\| - \|z - a\| \geq \mu - \left(\mu - \frac{3\varepsilon}{16}\right) = \frac{3\varepsilon}{16},$$

a contradiction to equality (3.1). Hence $y \in y_1\tilde{y}_1 \cup y_2\tilde{y}_2$. Since $z = y + (z - y) \in (y_1\tilde{y}_1 + \rho\widetilde{B}) \cup (y_2\tilde{y}_2 + \rho\widetilde{B})$ and $z \in \widetilde{Z}$ is arbitrary, it follows that $\widetilde{Z} \subset \widetilde{Z}_1 \cup \widetilde{Z}_2$. The reverse inclusion being trivial, we have $\widetilde{Z}_1 \cup \widetilde{Z}_2 = \widetilde{Z}$.

Now let $u \in u_1u_2$ be arbitrary. We have

$$d(u, Z) < d(u, Y_1), \tag{3.2}$$

where $Y_1 = X \cup (\tilde{y}_1, \tilde{y}_2) + \rho\widetilde{B}$. In order to prove (3.2) take $z_1 \in Z$ such that $\|y_1 - z_1\| \le \rho$. Clearly,

$$\|u - z_1\| \le \|u - u_1\| + \|u_1 - y_1\| + \|y_1 - z_1\| \le \|u_1 - u_2\| + \left(\mu - \frac{3\varepsilon}{8}\right) + \rho.$$

Furthermore, $\|u_1 - u_2\| = \varepsilon\|\tilde{y}_1 - \tilde{y}_2\|/(8\mu) = \varepsilon^2/(64\mu) < \varepsilon/8$, for $\varepsilon < \mu$. Hence $\|u - z_1\| < \mu - \varepsilon/4 + \rho$, which yields

$$d(u, Z) < \mu - \frac{\varepsilon}{4} + \rho. \tag{3.3}$$

On the other hand, let $y \in Y_1$ be arbitrary. Then $y = x + v$, for some $x \in X \cup \gamma(\tilde{y}_1, \tilde{y}_2)$ and $v \in \rho\widetilde{B}$. As $\|u - y\| \ge \|x - a\| - \|u - a\| - \|v\| \ge \mu - \varepsilon/8 - \rho$, we have

$$d(u, Y_1) \ge \mu - \frac{\varepsilon}{8} - \rho. \tag{3.4}$$

Combining (3.3) and (3.4) gives (3.2), since $\mu - \varepsilon/8 - \rho \ge \mu - \varepsilon/4 + \rho$.

By virtue of (3.3), each $z \in p_Z(u)$ satisfies $\|u - z\| < \mu - \varepsilon/4 + \rho = \|y_1 - a\| + \rho$; thus $z \in \widetilde{Z}$. Furthermore, (3.2) implies $p_Z(u) \cap Y_1 = \varnothing$, from which, in view of the definition of Y, it follows that $p_Z(u) \subset (y_1\tilde{y}_1 + \rho\widetilde{B}) \cup (y_2\tilde{y}_2 + \rho\widetilde{B})$. Hence

$$p_Z(u) \subset \widetilde{Z}_1 \cup \widetilde{Z}_2.$$

Let $z \in \widetilde{Z}_2$ be arbitrary. Let $y \in y_2\tilde{y}_2$ be such that $\|y - z\| \le \rho$. By Lemma 2, $\|y - u_1\| \ge \|y_2 - u_1\|$. Since $\|u_1 - z\| \ge \|u_1 - y\| - \rho \ge \|u_1 - y_2\| - \rho$, we have

$$\|u_1 - y_2\| \le d(u_1, \widetilde{Z}_2) + \rho. \tag{3.5}$$

Clearly $d(u_1, \widetilde{Z}_1) \le \|u_1 - y_1\| + \rho$, for $\widetilde{Z}_1 \cap B_{\mathbf{E}}(y_1, \rho) \neq \varnothing$. Moreover, $\|u_1 - y_1\| \le \|u_1 - y_2\| - 2\rho$, by virtue of the definition of ρ. Thus, $d(u_1, \widetilde{Z}_1) \le \|u_1 - y_2\| - \rho$. Combining this with (3.5) gives

$$d(u_1, \widetilde{Z}_1) \le d(u_1, \widetilde{Z}_2). \tag{3.6}$$

Analogously one can show that $d(u_2, \widetilde{Z}_1) \ge d(u_2, \widetilde{Z}_2)$. From this, (3.6) and the continuity of the function $u \to d(u, \widetilde{Z}_1) - d(u, \widetilde{Z}_2)$ from u_1u_2 to $\mathbf{R}$, it follows that there exists at least one point $u \in u_1u_2$ such that $d(u, \widetilde{Z}_1) = d(u, \widetilde{Z}_2)$. Furthermore, it is easy to see that $d(u, \widetilde{Z}_i) = d(u, Z)$, $i = 1, 2$, and so $p_{\widetilde{Z}_i}(u) \subset p_Z(u)$, $i = 1, 2$. Consequently, $p_Z(u)$ contains at least two different points, since $\widetilde{Z}_1 \cap \widetilde{Z}_2 = \varnothing$. Hence $Z \in \mathcal{N}_{a,k} \setminus \mathcal{N}_{a,k,r}$. Since Z is arbitrary in $B_{\mathcal{C}(\mathbf{E})}(Y, \rho)$, it follows that $B_{\mathcal{C}(\mathbf{E})}(Y, \rho) \cap \mathcal{N}_{a,k,r} = \varnothing$, completing the proof.

To complete the proof of Theorem 2 it remains to show that the following claim is true.

Claim. $\mathcal{N}_{a,r}$ is nowhere dense in $\mathcal{C}(\mathbf{E})$.

Proof. Indeed, let $X \in \mathcal{M}_{a,r}$ be arbitrary. Let $0 < \varepsilon < \min\{r, \nu\}$, where $\nu = e(a, X)$. Let $\tilde{y}_1 \in X$ and $\tilde{y}_2 \in \mathbf{E}$ satisfy $\|\tilde{y}_i - a\| = \nu$, $i = 1, 2$, and $\|\tilde{y}_i - \tilde{y}_2\| = \varepsilon/4$. For $i = 1, 2$, set

$$y_i = \tilde{y}_i + \frac{\varepsilon}{2\nu}(\tilde{y}_i - a), \qquad u_i = a + \frac{\varepsilon}{8\nu}(\tilde{y}_i - a),$$

and define $Y = X \cup y_1\tilde{y}_1 \cup y_2\tilde{y}_2 \cup \tilde{y}_1\tilde{y}_2$. Clearly, $Y \in \mathcal{C}(\mathbf{E})$ and $h(Y, X) \le 3\varepsilon/4$.

Let $Z \in B_{\mathcal{C}(\mathbf{E})}(Y, \rho)$ be arbitrary, where ρ is given by (3.1). Set

$$Z_i = Z \cap (\tilde{y}_i y_i + \rho \widetilde{B}), \qquad i = 1, 2.$$

It is evident that $Z_1, Z_2 \neq \varnothing$. Furthermore, $Z_1 \cap Z_2 = \varnothing$. Supposing otherwise, let $z \in Z_1 \cap Z_2$. Then for some $v_i \in y_i \tilde{y}_i$, $i = 1, 2$, we have $\|z - v_i\| \leq \rho$, $i = 1, 2$, and so $3\rho \leq \varepsilon(y_1 \tilde{y}_1, y_2 \tilde{y}_2) \leq \|v_1 - v_2\| \leq 2\rho$, a contradiction. Let $u \in u_1 u_2$ be arbitrary. Since each $y \in Y_1 = (X \cup \tilde{y}_1 \tilde{y}_2) + \rho \widetilde{B}$ satisfies $\|u - y\| \leq \|y - a\| + \|u - a\| \leq (\nu + \rho) + \varepsilon/8$, we have

$$e(u, Y_1) \leq \nu + \frac{\varepsilon}{8} + \rho. \tag{3.7}$$

Let $z \in Z \cap B_{\mathbf{E}}(y_1, \rho)$. Clearly,

$$\begin{aligned} e(u, Z) &\geq \|u - z\| \geq \|u - y_1\| - \|z - y_1\| \\ &> \|u_1 - y_1\| - \|u_1 - u\| - \rho \geq \|u_1 - y_1\| - \|u_1 - u_2\| - \rho. \end{aligned}$$

Moreover, $\|u_1 - y_1\| = \nu + 3\varepsilon/8$, and $\|u_1 - u_2\| = \varepsilon \|\tilde{y}_1 - \tilde{y}_2\|/(8\nu) = \varepsilon^2/(32\nu) < \varepsilon/8$, since $\varepsilon < \nu$. Hence,

$$e(u, Z) > \nu + \frac{\varepsilon}{4} - \rho. \tag{3.8}$$

By virtue of (3.7) and (3.8), observing that $\nu + \varepsilon/8 + \rho < \nu + \varepsilon/4 - \rho$, it follows that

$$e(u, Y_1) < e(u, Z).$$

This implies $q_Z(u) \cap Y_1 = \varnothing$ and so, from the definition of Y, $q_Z(u) \subset Z_1 \cup Z_2$.

Let $z \in Z_1$. Let $y \in y_1 \tilde{y}_1$ be such that $\|y - z\| \leq \rho$. Using (3.1), we have

$$\begin{aligned} \|u_1 - z\| &\leq \|u_1 - y\| + \|y - z\| \leq \|u_1 - y_1\| + \rho \leq \|u_1 - y_2\| - \rho \\ &\leq \|u_1 - v\| + \|v - y_2\| - \rho < \|u_1 - v\| \leq e(u_1, Z_2), \end{aligned}$$

for some $v \in Z_2 \cap B_{\mathbf{E}}(y_2, \rho)$, and so, since $z \in Z_1$ is arbitrary,

$$e(u_1, Z_1) \leq e(u_1, Z_2).$$

Likewise one can show that $e(u_2, Z_1) \geq e(u_2, Z_2)$. Then, arguing as in the proof of the preceding claim, one proves that $\mathcal{M}_{a,r}$ is nowhere dense in $\mathcal{C}(\mathbf{E})$. This completes the proof.

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