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CRUMPLED LAMINATIONS AND MANIFOLDS OF NONFINITE TYPE

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ABSTRACT. Using a group-theoretic construction due to Bestvina and Brady, we build $(n+1)$-manifolds W which admit partitions into closed, connected n-manifolds but which do not have finite homotopy type.

At the heart of this note is an example due to Bestvina and Brady [1] of an almost finitely presented group which is not finitely presented. Specifically, they describe a finitely presented group G with a perfect normal subgroup P such that G/P is not finitely presented (i.e., P fails to be the normal closure in G of a finite set); furthermore, P itself is expressed as an infinite free product $*_i P_i$ of finitely presented groups P_i, which happen to be pairwise isomorphic.

For each positive integer m let Γ_m denote $P_1 * P_2 * \cdots * P_m \subset P$, and let N_m denote the normal closure of Γ_m in G. Then

$$N_1 \subset N_2 \subset \cdots \subset N_m \subset N_{m+1} \subset \cdots$$

and $P = \bigcup N_m$. Set $G'_m = G/N_m$. Note the existence of natural projections $\psi_m : G'_m \to G'_{m+1}$; the direct limit of $\{\psi_m\}$ is G/P. Since $P = \bigcup N_m$ fails to be finitely generated as a normal subgroup, infinitely many of $\{\psi_m\}$ must have nontrivial kernel. This answers Questions 3.4 and 3.5 of [2]. We use it here to describe crumpled laminations $p : W \to \mathbb{R}$ on manifolds W which do not have finite homotopy type, answering Question 3.1 of [2] negatively, and illustrating the sharpness of the main result (Theorem 1.1) there.

Recall that a *crumpled lamination* on an $(n+1)$-manifold W is a closed map p of W to an interval J (possibly noncompact) such that each $p^{-1}(t)$ $(t \in J)$ is a closed, connected n-manifold.

Given a compact n-manifold M, $n \geq 5$, and a finitely generated, perfect subgroup H of $\pi_1(M)$, the mapping cylinder construction of [3] provides a map $f : M \to M'$ from M onto another n-manifold M' and a compact $(n+1)$-dimensional cobordism (W, M, M'), where the $(n+1)$-manifold W is obtained from the mapping cylinder of f by attaching a collar $M' \times [1, 2]$ to the obvious copy of M'; in addition, here the inclusion $M' \to W$ is a homotopy equivalence, $\pi_1(M')$ is isomorphic to the quotient of G by $N(H)$, the normal closure of H, and inclusion $M \to W$ induces the obvious projection $G \to G/N(H)$. Since W is determined as the disjoint union

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of $M \times [0,1)$ and $M' \times [1,2]$, it possesses a crumpled lamination derived from an obvious map $p : W \to [0,2]$ having n-manifolds as point preimages.

For $n \geq 5$ name a closed n-manifold M_0 for which $\pi_1(M_0) \cong G$. The mapping cylinder construction provides a laminated cobordism (W_1, M_0, M_1) such that $M_1 \to W_1$ is a homotopy equivalence, $\pi_1(M_1) \cong \pi_1(W_1) \cong G_1'$, and the inclusion $M_0 \to W_1$ induces the natural projection $\psi_0 : G \to G_1' = G/N_1$. Applying the construction recursively, we obtain successive laminated cobordisms (W_m, M_{m-1}, M_m) such that $M_m \to W_m$ is a homotopy equivalence, $\pi_1(M_m) \cong \pi_1(W_m) \cong G_m'$, and the inclusion $M_{m-1} \to W_m$ induces $\psi_{m-1} : G_{m-1}' \to G_m'$. We regard distinct W_i, W_j as intersecting only if $i = j \pm 1$ and then $W_i \cap W_{i+1} = M_i$. Consequently,

$$W = (M_0 \times (-\infty, 0)) \cup \left(\bigcup_{i=1}^{\infty} W_i \right)$$

is an $(n+1)$-manifold equipped with a lamination. It follows routinely that $\pi_1(W)$ is the direct limit of the inclusion-induced sequence

$$\left\{ \pi_1 \left((M_0 \times (-\infty, 0)) \cup \left(\bigcup_{i=1}^{m} W_i \right) \right) \to \pi_1 \left((M_0 \times (-\infty, 0)) \cup \left(\bigcup_{i=1}^{m+1} W_i \right) \right) \right\},$$

namely, G/P. Hence, W cannot be homotopy equivalent to a finite complex.

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