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EACH LOCALLY ONE-TO-ONE MAP FROM A CONTINUUM ONTO A TREE-LIKE CONTINUUM IS A HOMEOMORPHISM

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ABSTRACT. In 1977 T. Maćkowiak proved that each local homeomorphism from a continuum onto a tree-like continuum is a homeomorphism. Recently, J. Rogers proved that each locally one-to-one (not necessarily open) map from a hereditarily decomposable continuum onto a tree-like continuum is a homeomorphism, and this paper removes "hereditarily decomposable" from the hypothesis of Rogers' theorem.

It is not easy for a nice function to map onto a tree-like continuum without being a homeomorphism. T. Maćkowiak's classic result [3], proved in 1977, is that each local homeomorphism from a continuum onto a tree-like continuum is a homeomorphism. Local homeomorphisms are both open and locally one-to-one, and recently J. Rogers asked if "open" could be removed from the hypothesis of the Maćkowiak theorem. In [1] Rogers proved a special case, namely that if a locally one-to-one map that is not a homeomorphism is defined on a hereditarily decomposable continuum, then the image contains a continuum that is not unicoherent. Since all tree-like continua are hereditarily unicoherent, the image cannot be tree-like. These results come from the "complicated proof" found in [2] of Corollary 5.7 in [1]. We use the noun "map" to mean continuous function, and the term "continuum" to mean a connected, compact metric space.

The theorem to follow completes the task of removing "open" from the Maćkowiak theorem. The lemma that is proved first helps to organize the covers.

Definition. A finite collection of sets has a *tree-indexing* if its members can be labeled $\{L_1, L_2, ..., L_m\}$ so that the L_i are distinct and for each j from 2 to m, L_j intersects exactly one member of the set $\{L_1, L_2, ..., L_{j-1}\}$.

Tree-cover lemma. *A finite collection of open sets has a tree-indexing iff its nerve is a tree.*

Proof. Suppose $\{L_1, L_2, ..., L_m\}$ is a tree-indexing of a finite collection $\mathcal{V}$ of open sets. Then $\mathcal{V}$ is coherent since if $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, two disjoint subcollections whose unions do not intersect, and if $L_1 \in \mathcal{V}_1$, then the element of $\mathcal{V}_2$ with the smallest label fails the definition of a tree-indexing. Since $\mathcal{V}$ is coherent, its nerve is connected. Since no three elements in $\mathcal{V}$ can intersect, its nerve is a one-dimensional graph. If the nerve of $\mathcal{V}$ contains a simple closed curve, then the element of $\mathcal{V}$ with the

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largest label, that corresponds to some vertex in the simple closed curve, violates the tree-indexing definition. Therefore the nerve of $\mathcal{V}$ is a tree. Now, by way of contradiction, let $\mathcal{V}$ denote the smallest finite collection of open sets whose nerve is a tree but does not have a tree-indexing. Since the nerve of $\mathcal{V}$ is connected, the collection $\mathcal{V}$ must be coherent. Clearly, $\mathcal{V}$ must have more than two elements. Since the nerve of $\mathcal{V}$ has at least three vertices, the removal of an endpoint e, and the arc that connects e to the rest of the tree, leaves a tree. By our assumption, this means that the collection $\mathcal{V} \setminus \{V(e)\}$, where $V(e)$ denotes the open set in $\mathcal{V}$ that corresponds to e, has a tree-indexing $\{L_1, L_2, ..., L_m\}$. This generates a tree-indexing for $\mathcal{V}$ by labeling $V(e)$ as L_{m+1}. □

Theorem. *Every locally one-to-one map from a continuum onto a tree-like continuum is a homeomorphism.*

Proof. Suppose that h is a locally one-to-one map from a continuum X onto a tree-like continuum Y. It is clear that h must be finite-to-one, and there is a positive number ϵ such that if x and x' are points of X such that $h(x) = h(x')$, then $d(x, x') > 3\epsilon$. For each point y in Y, there is an open set $U(y)$ in Y containing y such that if $h^{-1}(y) = \{x_1, x_2, ..., x_n\}$, then $h^{-1}(U(y)) \subseteq \bigcup_{i=1}^{n} N_\epsilon(x_i)$ in X, where $N_\epsilon(x_i)$ denotes the ϵ neighborhood of x_i in X. Now, let $\mathcal{V}$ denote an open refinement of $\{U(y)|y \in Y\}$ that covers Y and whose nerve is a tree. By the tree-cover lemma, $\mathcal{V}$ can be written $\{L_1, L_2, ..., L_m\}$, satisfying the definition of a tree-indexing. For each $L_i \in \mathcal{V}$, let y_i denote a member of Y such that $L_i \subseteq U(y_i)$. Index the elements of $h^{-1}(y_i) = \{x_1, x_2, ..., x_k\}$, and for $j = 1, ..., k$, define $W(i, j) = h^{-1}(L_i) \cap N_\epsilon(x_j)$, if this set is non-empty. Note that for each relevant i and j, $W(i, j)$ is an open set in X of diameter less than ϵ, and h is one-to-one on $W(i, j)$. Now define $\mathcal{W}$ to be the set of these $W(i, j)$'s. Then $\mathcal{W}$ is an open covering of X.

3-link fact. *If $W(i_1, j_1)$, $W(i_2, j_2)$, and $W(i_3, j_3)$ are distinct elements of $\mathcal{W}$ such that $W(i_2, j_2)$ intersects each of the other two, then the integers $\{i_1, i_2, i_3\}$ are distinct.*

Let z_1 denote a point of $W(i_1, j_1) \cap W(i_2, j_2)$, let z_3 denote a point of $W(i_3, j_3) \cap W(i_2, j_2)$, and note that $d(z_1, z_3) < \epsilon$. First, suppose that $i_1 = i_2$. By construction, $W(i_1, j_1) \subseteq N_\epsilon(x_{j_1})$ and $W(i_1, j_2) \subseteq N_\epsilon(x_{j_2})$, where $h(x_{j_1}) = h(x_{j_2}) = y_{i_1}$. But $z_1 \in N_\epsilon(x_{j_1}) \cap N_\epsilon(x_{j_2})$ implies that $d(x_{j_1}, x_{j_2}) < 2\epsilon$, that is, $j_1 = j_2$. This is contrary to the fact that the W's are distinct. A similar contradiction occurs if $i_3 = i_2$. Secondly, suppose that $i_1 = i_3$. Again, by construction, $z_1 \in W(i_1, j_1) \subseteq N_\epsilon(x_{j_1})$ and $z_3 \in W(i_1, j_3) \subseteq N_\epsilon(x_{j_3})$, where $h(x_{j_1}) = h(x_{j_3}) = y_{i_1}$. Thus $d(x_{j_1}, x_{j_3}) > 3\epsilon$ if $j_1 \neq j_3$. But this is contrary to the fact that each of the following numbers is less than ϵ : $d(x_{j_1}, z_1)$, $d(z_1, z_3)$, and $d(x_{j_3}, z_3)$. This contradiction completes the proof of the 3-link fact.

Now, back to the proof of the theorem. If h is not a homeomorphism, then h is not one-to-one, so there exist two points x_1 and x_2 such that $h(x_1) = h(x_2)$. So $x_1 \in W(i, j)$ and $x_2 \in W(i, k)$ for some i and $j \neq k$, and there is a chain of elements from $\mathcal{W}$ with first link $W(i, j)$ and last link $W(i, k)$. Let $C = \{W(k_1, n_1), W(k_2, n_2), ..., W(k_m, n_m)\}$ denote a chain in $\mathcal{W}$ of shortest length such that $k_1 = k_m$. By the 3-link fact, $m > 3$. The indexing on C is understood to be the usual chain indexing, where the links are distinct and $W(k_i, n_i)$ intersects $W(k_j, n_j)$ iff $|i - j| \leq 1$. Let k_j be the smallest integer in $\{k_1, k_{2,}, ..., k_m\}$, where we use $j = 1$ if the smallest is $k_1 = k_m$. Then $k_{j+1} > k_j$ and $k_{j+2} > k_{j+1}$. To see this second inequality, note that in $\mathcal{V}$, $L_{k_{j+1}}$ intersects L_{k_j} since $W(k_{j+1}, n_{j+1})$ intersects $W(k_j, n_j)$, so L_{k_j} is the only element of $\mathcal{V}$ with lower subscript that $L_{k_{j+1}}$ intersects. This means that, since $L_{k_{j+1}}$ also intersects $L_{k_{j+2}}$, the subscript k_{j+2} must be greater than k_{j+1}. If we continue in this way we can establish the fact that $k_j < k_{j+1} < k_{j+2} < ... < k_m = k_1 < k_2 < ... < k_{j-1}$. Note that when we "turn the corner" we use the fact that $k_2 \neq k_{m-1}$, which follows since $m > 3$. The final contradiction is that the last link $L_{k_{j-1}}$ intersects both of the lower indexed links $L_{k_{j-2}}$ and L_{k_j}. □

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