EMBEDDING OF A BANACH ALGEBRA $A$ INTO $L(A')$

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Abstract. Given a Banach space $X$, we have shown in 1994 that a product can be defined on it in such a way that the resulting Banach algebra is isomorphic to a compact subalgebra of the algebra $L(X')$ of all bounded linear operators on the topological dual $X'$ of $X$. Our purpose here is to prove that, more generally, any Banach algebra $A$ admitting a left approximate identity, is isomorphic to a subalgebra of $L(A')$, the isomorphism being isometric, provided the approximate identity is bounded by 1. As a consequence, we get a factorization through $L(A')$, of the elements in $A'$. An abstract $C^*$-algebra is isometrically isomorphic to a uniformly closed subalgebra of $L(H)$, the algebra of all bounded linear operators on some Hilbert space $H$. When $A$ is no longer a $C^*$-algebra, but some Banach algebra with a left approximate identity, we still get an isomorphism from $A$ into an algebra of bounded linear operators on some Banach space $E$ which is no longer a Hilbert space. More precisely, $E$ is the topological dual $A'$ of $A$. When the approximate identity is a subset of the unit sphere of $A$, we get a similar identification of $A$ as in the case of a $C^*$-algebra, in the following way:

Any Banach algebra $A$ with a left approximate identity bounded by 1 is isometrically isomorphic to a uniformly closed subalgebra of $L(A')$.

When $A$ is a commutative Banach *-algebra, then the isomorphism is an involution-preserving map.

We mainly derive from the above identification of $A$ with a subalgebra of $L(A')$, a particular factorization of the elements of $A'$, through $L(A')$. Let $A$ and $B$ be Banach spaces each of which is endowed with a linear involution denoted by “$*$” and “$•$” respectively. We consider:

1. The linear involution “$⊗$” defined on the space $L(A,B)$ by:

$$P⊗(a) = P(a^*)^* \quad (P \in L(A,B); a \in A).$$

2. The linear involution “$κ$” defined on the space $A'$ by:

$$T^κ(x) = T(x^*) \quad (x \in A; T \in A').$$

($T^κ(x) = \overline{T(x)}$, if $A$ does not have an involution, where the bar denotes complex conjugation).

The following is the main result of the paper.

Theorem 1. Let $A$ be a Banach algebra with a left approximate identity $(u_α)_{α \in I}$ bounded by 1. Then
(i) There exists an isometric isomorphism \( \sigma \) from \( A \) onto a uniformly closed subalgebra of \( L(A') \).

(ii) If \( A \) is a commutative Banach \(*\)-algebra, then the isometry \( \sigma \) is \(*\)-map in the following sense: \( \sigma(x^*) = \sigma(x)^\circ \), \( \forall x \in A \).

Proof. For all \( x, y, \in A \), and all \( T \in A' \), we set:

\[
(3) \quad (\sigma(x)(T))(y) = T(yx).
\]

Then easy computations show that the map \( x \mapsto \sigma(x) \) from \( A \) into \( L(A') \) induced from relation (3) above is a homomorphism. Moreover, \( \sigma \) is one-to-one as follows: if \( x, y \in A \) are such that \( \sigma(x) = \sigma(y) \), then for all \( z \in A \) and \( T \in A' \), we have \( T(zx) = T(zy) \). But since \( A \) admits a left approximate identity \((u_\alpha)_{\alpha \in I}\), we get

\[
(4) \quad T(u_\alpha x) = T(u_\alpha y) \quad (\alpha \in I, T \in A'),
\]

that is,

\[
\|T(x - y)\| = \|T(u_\alpha(x - y)) - T(x - y)\| \\
\leq \|T\| \|u_\alpha(x - y) - (x - y)\|,
\]

from which it follows that

\[
(5) \quad T(x) = T(y) \quad \forall T \in A',
\]

and therefore, \( x = y \).

Moreover, for all \( x \in A \), it is easily checked that

\[
(6) \quad \|\sigma(x)\| \leq \|x\|.
\]

Conversely, if \( A_1 \) (resp. \( A'_1 \)) denotes the closed unit ball of \( A \) (resp. \( A' \)), then given \( x \in A \), we have, for all \( \alpha \in I \):

\[
\|u_\alpha x\| \leq \operatorname{Sup}\{\|yx\|; y \in A_1\} \\
\leq \operatorname{Sup}\{\operatorname{Sup}\{|T(yx)|; T \in A'_1\}; y \in A_1\} \\
\leq \operatorname{Sup}\{\operatorname{Sup}\{|(\sigma(x)T)(y)|; T \in A'_1\}; y \in A_1\}.
\]

It follows, when \( \alpha \) tends to infinity, that

\[
(7) \quad \|x\| \leq \|\sigma(x)\|,
\]

which, together with relation (6), shows that \( \sigma \) is isometric. The range \( \sigma(A) \) of \( \sigma \) is clearly closed under the uniform norm of \( L(A') \).

(ii) If \( A \) is in addition a commutative \(*\)-algebra, then \( \sigma \) is an involution-preserving map, for if \( x, y \in A \) and \( T \in A' \), then

\[
(\sigma(x^*)(T))(y) = T(y x^*) = T(x^* y) \quad (A \text{ is commutative})
\]

\[
= T(y^* x) = (\sigma(x)(T^\circ))(y^*)
\]

\[
= (\sigma(x)(T^\circ))^\circ(y) = (|\sigma(x)|^\circ(T))(y),
\]

from which it follows that:

\[
(8) \quad \sigma(x^*) = [\sigma(x)]^\circ,
\]

and the proof is complete. \( \square \)

**Corollary 2.** If \( A \) is a commutative \( C^* \)-algebra, then \( \sigma(A) \) endowed with the involution \( \otimes \) is a commutative \( C^* \)-algebra.
Proof. Each $C^*$-algebra admits a two-sided approximate identity bounded by 1. According to Theorem 1 (ii) above, $\sigma(A)$ is a commutative $\otimes$-sub-algebra of $L(A')$ satisfying, for all $x \in A$:

$$\|\sigma(x) \otimes \sigma(x)\| = \|\sigma(x^* x)\| = \|x^* x\| = \|x\|^2 = \|\sigma(x)\|^2.$$  \[ \square \]

Corollary 3. Let $G$ be a topological locally compact group and $C_0(G)$ the algebra of all continuous complex functions on $G$ vanishing at infinity.

(i) If $M^1(G)$ denotes the space of all bounded Radon measures on $G$, then $L(M^1(G))$ contains an isometric image of $C_0(G)$.

(ii) If $G$ is abelian, then $L^1(G)$ is isometrically isomorphic to a subalgebra of $L(L^\infty(G))$.

Proof. (i) $A = C_0(G)$ endowed with the sup-norm and the involution defined by complex conjugation is a commutative $C^*$-algebra. Hence according to Theorem 1 and Corollary 2, $A$ may be identified with a commutative $C^*$-algebra in $L(A')$. But the Riesz representation theorem ensures that $A'$ is isometrically isomorphic to $M^1(G)$.

(ii) If $G$ is abelian, the group algebra $L^1(G)$ endowed with the usual convolution of functions is a commutative Banach $*$-algebra with a two-sided approximate identity bounded by 1. Hence Theorem 1 holds, and the result follows. \[ \square \]

We shall need the following lemma for the next result.

Lemma 4. Let $X,Y,Z$ be Banach spaces, and $P,Q$ be linear operators from $X$ into $Y$ and from $X$ into $Z$ respectively, such that $\text{Ker}(P)$ and $\text{Ker}(Q)$ satisfy $\text{Ker}(P) \subset \text{Ker}(Q)$. Then there exists a linear map $S$ from the range $R(P)$ of $P$ into $Z$ satisfying $Q = SP$.

Proof. Straightforward. \[ \square \]

We have the following particular decomposition of the elements of $A'$:

Theorem 5. Let $A$ be a Banach algebra with a left approximate identity bounded by 1. Then, each $T \in A'$ admits the following factorization:

$$T = \Phi \Theta \sigma, \quad (9)$$

where

$\sigma$ is the isometry in Theorem 1,

$\Theta$ is a continuous linear map from $\sigma(A) \subset L(A')$ into $A'$, and

$\Phi$ is an element in the bidual $A''$ of $A$.

Proof. Let $(u_\alpha)_{\alpha \in I}$ be a left approximate identity. For each $T \in A'$, we define the map $\Psi_T$ as follows:

$$x \mapsto \psi_T(x) = \sigma(x)(T) \quad (x \in A). \quad (10)$$

$\psi_T$ is well-defined, linear and continuous ($\|\psi_T\| \leq \|T\|$), and takes its values in $A'$. Moreover, using the existence of a (bounded) left approximate identity in $A$, one shows that

$$\text{Ker}(\psi_T) \subset \text{Ker}(T) \quad (11)$$

$$[\sigma(x)(T)](\cdot) = T(\cdot x).$$

But then, according to Lemma 4, there exists a linear map $\varphi$ from the range $\psi_T(A)$ of $\psi_T$ onto $\mathbb{C}$ such that $T = \varphi \psi_T$. $\varphi$ is continuous on
Moreover, the range, say \( \rho(x)(T) \), of \( \rho \) in \( \mathcal{K}(X') \), consists of rank-one operators on \( X' \). Namely,

\[
\rho(x)(T) = T(xy) = F(y)T(x), \quad x, y \in X.
\]

(16) Moreover, the range, say \( A_\rho(X, F) \) of \( \rho \) in \( \mathcal{K}(X') \), consists of rank-one operators on \( X' \). Namely,

\[
\rho(x)(T) = T(xy) = \hat{x}(T)F, \quad x \in X, T \in X',
\]

where \( x \mapsto \hat{x} \) denotes the natural linear isometry of \( X \) into \( X'' \).
Remark 8. It turns out that Theorem 6 above is a particular case of Theorem 1, according to the following:

**Proposition 9.** The Banach algebra $A(X, F)$ in Theorem 6 admits infinitely many left approximate identities bounded by 1. Moreover precisely, $(u_n)_{n \in I}$ is a left approximate identity for $A(X, F)$ if and only if one has:

$$\lim_{\alpha \to +\infty} F(u_\alpha) = 1.$$  \hspace{1cm} (17)

In particular, if $u \in X$ is such that $F(u) = 1$, then the sequence $(u_n)_{n \geq 1}$ such that $u_n = z_n u$, where $(z_n)_{n \geq 1}$ is any complex sequence in the unit disk ($\|z_n\| \leq 1; n \geq 1$) satisfying $\lim_{n \to +\infty} z_n = 1$, is a countable left approximate identity for $A(X, F)$, bounded by 1.

**Proof.** Straightforward. \hfill $\square$

We close the sequel with the following two properties of $A_{\rho}(X, F)$:

**Proposition 10.** (i) For all $x \in X$ and $P \in L(X')$, the following condition holds:

$$P \circ \rho(x)(\cdot) = \hat{x}(\cdot) P(F).$$  \hspace{1cm} (18)

(ii) For all $x \in X$ and $P \in L(X')$, there exists, for each $T \in X'$, some $z_{x,T,P} \in X$ (depending on $x, P$ and $T$) such that:

$$\rho(x) \circ P |_{C T} = \rho(z_{x,T,P}),$$

where $(\rho(x) \circ P)|_{C T}$ stands for the restriction of $(\rho(x) \circ P)$ to the one-dimensional subspace $C T$ of $X'$.

**Proof.** (i) is obvious.

(ii) Let $x \in X$ and $P \in L(X')$. Then, for each $T \in X'$ and $\lambda \in \mathbb{C}$, we have $(\rho(x) \circ P)(\lambda T) = \lambda P(T)(x) F$. But we may choose $z_{x,T,P} \in X$ such that $P(T)(x) = T(z_{x,T,P})$, and it follows that

$$\rho(x) \circ P)(\lambda T) = \lambda T(z_{x,T,P}) F = \lambda \hat{z}_{x,T,P}(T) F = \lambda \rho(z_{x,T,P})(T) = \rho(z_{x,T,P})(\lambda T).$$

\hfill $\square$

**Corollary 11.** $A_{\rho}(X, F)$ is a proper subspace of $K(X')$.

**Proof.** $A_{\rho}(X, F)$ is clearly not reduced to $\{0\}$. Since $K(X')$ is a two-sided ideal of $L(X')$, the assumption $A_{\rho}(X, F) = K(X')$ would imply that, given $x \in X$ and $P \in L(X')$, there exists some $u_{x,P} \in X$ satisfying:

$$P \circ \rho(x) = \rho(u_{x,P}).$$  \hspace{1cm} (20)

Let $\text{id}_{L(X')}$ denote the identity map on $L(X')$. We may assume that $P \notin \mathbb{C} \text{id}_{L(X')}$ in relation (20) above. According to relation (15), we get from relation (20) that

$$T(x) P(F)(y) = T(u_{x,P}) F(y)$$  \hspace{1cm} (21)

for all $y \in X$ and for some nonzero $T$ in $X'$.

Now, for all $x \in X$ and for any $y$ in the kernel $\text{Ker}(F)$ of $F$, relation (21) above implies:

$$0 = T(x) P(F)(y) \quad (x \in X, P \in L(X')).$$
and hence \( y \in \ker(P(F)) \). Therefore, \( \ker(F) \subset \ker(P(F)) \) and the following holds:

(22) \( P(F) = \lambda_F F, \quad \lambda_F \in \mathbb{C} \).

But this forces \( P \) to satisfy \( P \in \mathbb{C} \text{id}_{L(X')} \), in contradiction with the choice of \( P \), and the conclusion follows. \( \square \)

**Remark 12.** In the proof of Corollary 11, we have in fact showed that \( A_\rho(X,F) \) cannot be a left ideal of \( L(X') \). Whether \( A_\rho(X,F) \) is a right ideal of \( L(X') \) or not, is a question to which we do not yet have an answer.

**References**


