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WEAK COMPACTNESS IN $L^1(\mu, X)$

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ABSTRACT. We characterize weak compactness and weak conditional compactness of subsets of $L^1(\mu, X)$ in terms of regular methods of summability. We also study when these results still hold using only convergence in the sense of Cesàro.

1. INTRODUCTION

Let X be a Banach space and $L^1(\mu, X)$ the Banach space of (equivalence classes of) Lebesgue–Bochner integrable functions over a finite measure space (Ω, Σ, μ). Ülger [15] proved that a bounded subset A of $L^\infty(\mu, X)$ is weakly relatively compact in $L^1(\mu, X)$ if and only if, given any sequence $(f_n) \subset A$, there exists a sequence (g_n), with $g_n \in co\{f_k : \ k \geq n\}$, such that $(g_n(\omega))$ is weakly convergent in X for almost every $\omega \in \Omega$, solving in this way a long–standing open problem (see Chapter IV of [6] and references in [15]). Later, Diestel, Ruess and Schachermayer [7] removed the restriction of L^∞–boundedness and gave a more elementary proof.

In this paper, we describe the above "convex compactness condition" in terms of regular methods of summability in X. This provides new characterizations of weak compactness as well as weak conditional compactness in $L^1(\mu, X)$. In fact, for a wide class of Banach spaces, it is enough to consider only convergence in the sense of Cesàro. We refer the reader to the monographs of Diestel [4], Diestel and Uhl [6] or Dunford and Schwartz [8] for the terminology and notation in this paper.

2. RESULTS

The main task of summability theory is to assign a limit to a divergent sequence or a sum to a divergent series, and the most common way to do this is to use an infinite matrix. An important property of these matrices is the so–called regularity.

We recall that an infinite matrix $T = (t_{n,m})$ of scalars is said to be a regular method of summability in a Banach space X if, for every convergent sequence (x_n) in X, the sequence $x_n^T = \sum_{m=1}^{\infty} t_{n,m} x_m$ exists for each $n \in \mathbb{N}$ and it is convergent to the same limit as (x_n). T is said to be positive if $t_{n,m} \geq 0$, for all $m, n \in \mathbb{N}$. Regular matrices are characterized by the vectorial version of the classical Silverman–Toeplitz Theorem (see, for instance [3, Theorem 1, p. 96]). In particular, they do not depend on the particular Banach space X.

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Theorem A (Silverman–Toeplitz). *A scalar infinite matrix $T = (t_{n,m})$ is a regular summability method in X if and only if it satisfies the following three conditions:*

(A) $\sum_{m=1}^{\infty} |t_{n,m}| \leq M$, *for some $M > 0$ and for all $n \in \mathbb{N}$.*
(B) $\lim_n t_{n,m} = 0$ *for all $m \in \mathbb{N}$.*
(C) $\lim_n \sum_{m=1}^{\infty} t_{n,m} = 1$.

Note that, if we drop a finite or an infinite number of rows or if we add a finite or infinite number of columns of zeroes in a regular method of summability, we still have a regular method of summability.

We begin by showing how these methods can be employed to analyze weak compactness and weak conditional compactness in a Banach space. In this line, we present a slight refinement of a result of Watermann [16], slight in the sense that we do not impose the positivity of the matrix. We also think that our proof is more elementary.

Theorem 1. *Let A be a subset of X. Then, the following are equivalent:*

(1) *A is weakly relatively compact.*
(2) *A is bounded, and for each sequence $(x_n) \subset A$, there is a regular method of summability T such that x_n^T is norm convergent.*
(3) *A is bounded, and for each sequence $(x_n) \subset A$, there is a regular method of summability T such that x_n^T is weakly convergent.*

Proof. (1)⇒(2) Assume that $(x_n) \subset A$. Applying Eberlein–Shmul'yan's Theorem [4, p. 18] and Mazur's Theorem [4, Chapter II, Corollary 2], we can find an increasing sequence of natural numbers $0 = p_0 < p_1 < p_2 < \ldots < p_n < \ldots$ such that the sequence of convex combinations of the x_n, $x_n' = \sum_{k=p_{n-1}+1}^{p_n} \lambda_{n,k} x_k$, $n = 1, 2, \ldots$, converges in norm. Hence, the matrix $T = (t_{n,m})$ defined as $t_{n,m} = \lambda_{n,m}$, when $p_{n-1} < m \leq p_n$ $(n \in \mathbb{N})$ and zero otherwise, is a regular method of summability. Finally, note that $x_n' = x_n^T$ for all $n \in \mathbb{N}$.

(2)⇒(3) is clear. (3)⇒(1) According to a theorem due to Grothendieck ([10, §24, 3.(8)], [7]), it is sufficient to prove that A has the interchangeable double limit property. Thus, take $(x_n) \subset A$, (x_m^*) contained in the closed unit ball of X^*, and assume that the limits $\lim_m \lim_n \langle x_m^*, x_n \rangle$, $\lim_n \lim_m \langle x_m^*, x_n \rangle$ exist.

By (3), we have a regular method of summability T such that (x_n^T) is weakly convergent to some $x \in X$. Therefore, if x^* is a weak* cluster point of (x_m^*) in X^*, we have

$$\begin{aligned}\lim_{n\to\infty} \lim_{m\to\infty} \langle x_m^*, x_n \rangle &= \lim_{n\to\infty} \langle x^*, x_n \rangle = \lim_{n\to\infty} \sum_{k=1}^{\infty} t_{n,k} \langle x^*, x_k \rangle \\ &= \lim_{n\to\infty} \langle x^*, \sum_{k=1}^{\infty} t_{n,k} x_k \rangle = \lim_{n\to\infty} \langle x^*, x_n^T \rangle = \langle x^*, x \rangle,\end{aligned}$$

and, on the other hand,

$$\begin{aligned}\lim_{m\to\infty} \lim_{n\to\infty} \langle x_m^*, x_n \rangle &= \lim_{m\to\infty} \lim_{n\to\infty} \sum_{k=1}^{\infty} t_{n,k} \langle x_m^*, x_k \rangle \\ &= \lim_{m\to\infty} \lim_{n\to\infty} \langle x_m^*, \sum_{k=1}^{\infty} t_{n,k} x_k \rangle = \lim_{m\to\infty} \lim_{n\to\infty} \langle x_m^*, x_n^T \rangle \\ &= \lim_{m\to\infty} \langle x_m^*, x \rangle = \langle x^*, x \rangle.\end{aligned}$$

□

A consequence of this theorem is that weak sequential convergence can be characterized in terms of norm sequential convergence.

Corollary 1. *Let (x_n) be a bounded sequence in X and $x \in X$. Then, weak–$\lim_n x_n = x$ if and only if for each subsequence (x_{n_k}) there is a regular method of summability T such that norm–$\lim_k x^T_{n_k} = x$.*

Proof. ($\Rightarrow$) It follows from Mazur's Theorem. ($\Leftarrow$) By Theorem 1, we have that the subset $\{x_n : n \in \mathbb{N}\}$ is weakly relatively compact. Hence, any subsequence (x_{n_k}) has a further subsequence, which we still denote (x_{n_k}), weakly convergent to some $y \in X$. On the other hand, there is also a regular method of summability T such that $\lim_k x^T_{n_k} = x$. Hence $y = x$. Finally, note that (x_n) is weakly convergent to x if and only if any subsequence of (x_n) contains a further subsequence which converges weakly to x. □

Theorem 2. *Let A be a subset of X. Then, A is weakly conditionally compact if and only if A is bounded and for each sequence $(x_n) \subset A$, there is a regular method of summability method T such that x^T_n is weakly Cauchy.*

Proof. ($\Rightarrow$) Suppose that $(x_n) \subset A$. Since A is weakly conditionally compact, we can obtain a weakly Cauchy subsequence (x_{n_k}). Then, the matrix $T = (t_{n,m})$ defined as $t_{k,n_k} = 1$ ($k \in \mathbb{N}$) and zero otherwise is a regular method of summability. Finally, note that $x^T_k = x_{n_k}$, for all $k \in \mathbb{N}$.

($\Leftarrow$) Assume that $(x_n) \subset A$ is a bounded sequence without any weakly Cauchy subsequence. According to Rosenthal's ℓ_1–Theorem [4, p. 201], (x_n) has a subsequence, that we still denote (x_n), equivalent to the unit basis of ℓ_1. Then, the closed linear span of (x_n) in X can be denoted by Y^*, for some Banach space Y isomorphic to c_0. Of course, Y^* is weakly sequentially complete [4, Chapter VII, Theorem 12].

Moreover, we know that there is a regular method of summability T such that x^T_n is weakly Cauchy. Bearing in mind Theorem A, we have that condition (A) means that $x^T_n \in Y^*$, for all $n \in \mathbb{N}$, and therefore, x^T_n is $\sigma(Y^*, Y^{**})$–convergent, condition (B) means that x^T_n is $\sigma(Y^*, Y)$–convergent to zero and condition (C) means that x^T_n is not $\sigma(Y^*, Y^{**})$–convergent to zero. Hence, we have obtained a contradiction. □

In order to prove our results about Lebesgue–Bochner integrable functions, we need the following fact which has been observed (more or less explicitely) in some papers [7], [14], [15]. We sketch a proof for the sake of completeness.

Lemma 1. *Let (f_n) be a uniformly integrable bounded sequence in $L^1(\mu, X)$.*

1. *If (f_n) is weakly convergent almost surely to a function $f \in L^1(\mu, X)$, then (f_n) is weakly convergent to f in $L^1(\mu, X)$.*
2. *If (f_n) is weakly Cauchy almost surely, then (f_n) is weakly Cauchy in $L^1(\mu, X)$.*

Proof. (1) We may, and do, assume that $L^1(\mu, X)$ is separable. Then, continuous linear functionals on $L^1(\mu, X)$ are represented by weak*–measurable essentially bounded functions from Ω to X^*. Take $f^* : \Omega \to X^*$, $f^* \in L^1(\mu, X)^*$. Then, uniform integrability of the sequence $(\langle f^*, f_n \rangle)$, the fact that $\langle f^*(\cdot), f_n(\cdot) \rangle \to \langle f^*(\cdot), f(\cdot) \rangle$ almost surely in X and Vitali's Convergence Theorem [6, p. 35] yield our assertion.

(2) It is a consequence of (1) and the fact that a sequence (x_n) in an arbitrary Banach space is weakly Cauchy if and only if for any increasing sequences of natural numbers (n_k), (m_k) weak–$\lim_k(x_{n_k} - x_{m_k}) = 0$. □

The proof of our next result is inspired in that of [7, Theorem 2.1].

Theorem 3. *Let A be a bounded subset of $L^1(\mu, X)$. Then, the following are equivalent:*

(1) *A is weakly relatively compact.*
(2) *A is uniformly integrable, and, for each sequence $(f_n) \subset A$, there is a regular method of summability T such that $(f_n^T(\omega))$ is norm convergent for almost every $\omega \in \Omega$.*
(3) *A is uniformly integrable, and, for each sequence $(f_n) \subset A$, there is a regular method of summability T such that $(f_n^T(\omega))$ is weakly convergent for almost every $\omega \in \Omega$.*

Proof. (1)⇒(2) If (1) holds, then A is uniformly integrable by [6, p. 104]. On the other hand, suppose that $(f_n) \subset A$. By Theorem 1, there is a regular method of summability T such that f_n^T is $\|\cdot\|_1$–convergent to some $f \in L^1(\mu, X)$. Hence, a subsequence of (f_n^T) converges to f pointwise almost surely in the norm of X. Dropping suitable rows from T, we finally obtain a regular method of summability T' such that $(f_n^{T'})$ is norm convergent almost surely.

(2)⇒(3) being obvious, it remains to prove (3)⇒(1). The idea is to apply Theorem 1 again. Thus, given a sequence (f_n) contained in A, we can find a regular method of summability $T = (t_{n,m})$ and a null measure subset $E \in \Sigma$ such that $(f_n^T(\omega))$ is weakly convergent for $\omega \in \Omega \setminus E$. Define $f : \Omega \to X$ by $f(\omega) =$ weak–$\lim_n f_n^T(\omega)$ for $\omega \in \Omega \setminus E$ and by zero, otherwise. Clearly, f is essentially separably valued and weakly measurable, hence strongly measurable by Pettis's Measurability Theorem [4, p. 25]. Moreover, using Fatou's Lemma [8, III.6.19], boundedness of A and condition (A) from Theorem A,

$$\int \|f\| d\mu \le \int \liminf \|f_n^T\| d\mu \le \liminf \int \|f_n^T\| d\mu$$
$$\le \sup_n \sum_{m=1}^{\infty} |t_{n,m}| \sup_n \int \|f_n\| d\mu < \infty.$$

Since A is uniformly integrable, we deduce from condition (A) of Theorem A that (f_n^T) is also uniformly integrable. The proof finishes by applying Lemma 1(1) to the sequence (f_n^T). □

From this theorem, we can deduce a pointwise version of Corollary 1.

Corollary 2. *Let (f_n) be a bounded uniformly integrable sequence in $L^1(\mu, X)$ and $f \in L^1(\mu, X)$. Then, weak–$\lim_n f_n = f$ if and only if for each subsequence (f_{n_k}), there is a regular method of summability T such that norm–$\lim_k f_{n_k}^T(\omega) = f(\omega)$, for almost every $\omega \in \Omega$.*

The next result is the version of Theorem 2 for weak conditional compactness.

Theorem 4. *Let A be a bounded subset of $L^1(\mu, X)$. Then, A is weakly conditionally compact if and only if A is uniformly integrable and, for each sequence $(f_n) \subset A$, there is a regular method of summability T such that $(f_n^T(\omega))$ is weakly Cauchy for almost every $\omega \in \Omega$.*

Proof. ($\Rightarrow$) A is uniformly integrable by [6, p. 104]. On the other hand, given a sequence $(f_n) \subset A$, we can extract a weakly Cauchy subsequence (f_{n_k}) in $L^1(\mu, X)$. According to [14], we can write $f_{n_k} = g_{n_k} + h_{n_k}$, where $(h_{n_k})_k$ tends weakly to zero in $L^1(\mu, X)$ and $(g_{n_k})_k$ is weakly Cauchy almost surely. Let E_1 be the exceptional subset of Ω associated to (g_{n_k}).

Applying Theorem 3(3) to (h_{n_k}), we obtain a regular method of summability T and a null measure subset $E_2 \in \Sigma$ such that $(h^T_{n_k}(\omega))$ is weakly convergent for $\omega \in \Omega \setminus E_2$. It is clear that $(g^T_{n_k}(\omega))$ is weakly Cauchy for $\omega \in \Omega \setminus E_1$, and, therefore, $(f^T_{n_k}(\omega))$ is weakly Cauchy for all $\omega \in \Omega \setminus (E_1 \cup E_2)$. Adding some columns of zeroes to T, we still have a regular method of summability T' such that $(f^{T'}_n)$ is weakly Cauchy almost surely.

($\Leftarrow$) We shall apply Theorem 2. Given a sequence (f_n) contained in A, we can find a regular method of summability T such that (f^T_n) is weakly Cauchy almost surely. Since A is uniformly integrable, we deduce from condition (A) from Theorem A that (f^T_n) is also uniformly integrable. Now, by Lemma 1(2), we obtain that (f^T_n) is weakly Cauchy in $L^1(\mu, X)$. Finally, apply Theorem 2. □

Usual definitions of weak compactness and weak conditional compactness in Banach spaces fit with the scheme of this paper, because the identity matrix and those obtained from it by adding columns of zeroes are regular methods of summability. In what follows, we are going to deal with arithmetic means, i.e., regular methods of summability associated to Cesàro matrix $C = (c_{n,m})$ ($c_{n,m} = 1/n$ for $1 \leq m \leq n$ and zero otherwise).

In the following three corollaries, we show analogous results to Theorems 1 and 2, using convergence in Cesàro sense. We recall that a Banach space X is said to have the Banach–Saks property if every bounded sequence in X has a subsequence whose arithmetic means are norm convergent. It is worth mentioning that there are reflexive Banach spaces without the Banach–Saks property [5, p. 84]. The first result follows directly from Theorem 1(2) and was known to Nishiura and Watermann [5, p. 83].

Corollary 3. *Every Banach space with the Banach–Saks property is reflexive.*

Corollary 4. *X is reflexive if and only if every bounded sequence in X admits a subsequence whose arithmetic means are weakly convergent.*

Proof. On the one hand, use the weak sequential compactness of the closed unit ball of X, and, on the other hand, consider Theorem 1(3). □

Corollary 5. *X has no copy of ℓ_1 if and only if every bounded sequence in X admits a subsequence whose arithmetic means are weakly Cauchy.*

Proof. On the one hand, use Rosenthal's Theorem, and, on the other hand, consider Theorem 2. □

In order to give the Cesàro results for spaces of Lebesgue–Bochner integrable functions, we need some lemmas which can be of independent interest.

Lemma 2. *Let (f_n) be a sequence in $L^1(\mu, X)$. If (f_n) is weakly convergent to f in $L^1(\mu, X)$ and (f_n) is weakly Cauchy almost surely, then (f_n) is weakly convergent to f almost surely.*

Proof. First of all, we state the following fact that can be proved as in Theorem 1: if (x_n) is a weakly Cauchy sequence in X and T is a regular method of summability, then we have

(1) (x_n^T) is weakly Cauchy and $\lim_n\langle x^*, x_n\rangle = \lim_n\langle x^*, x_n^T\rangle$ for all $x^* \in X^*$.
(2) (x_n) is weakly convergent to x if and only if (x_n^T) is weakly convergent to x.

Now, suppose that, for all $x^* \in X^*$ and for all $\omega \in \Omega \setminus E_1,\ \mu(E_1) = 0$, the limit $\lim_n\langle x^*, f_n(\omega)\rangle$ exists. Since (f_n) is weakly convergent in $L^1(\mu, X)$, by Theorem 3(3), there is a regular method of summability T such that $(f_n^T(\omega))$ is weakly convergent to $f(\omega) \in X$, for all $\omega \in \Omega \setminus E_2,\ \mu(E_2) = 0$.

Therefore, for all $\omega \in \Omega \setminus (E_1 \cup E_2)$ and for all $x^* \in X^*$ and bearing in mind the fact above, we have

$$\langle x^*, f(\omega)\rangle = \lim_n\langle x^*, f_n^T(\omega)\rangle = \lim_n\langle x^*, f_n(\omega)\rangle.$$

□

The next lemma has its roots in a result due to Komlós [9] about the scalar–valued space $L^1(\mu)$.

Theorem B (Komlós). *For every bounded sequence (f_n) in $L^1(\mu)$ there exist a subsequence (f_n') of (f_n) and f in $L^1(\mu)$ such that*

$$\frac{1}{k}\sum_{n=1}^{k} f_n^{''} \to f \quad \textit{almost everywhere}$$

for each subsequence $(f_n^{''})$ of $(f_n^{'})$.

This theorem has been extended by Balder [1] to $L^1(\mu, X)$, when X is reflexive. Our following results enlarge the range of Banach spaces to which it is applicable. We recall that a subspace N of X^* is said to be norming for X if

$$\|x\| = \{\langle x, y^*\rangle :\ y^* \in B_N\}, \quad \text{for all } x \in X.$$

In particular, (X, N) is a dual pair.

Lemma 3. *Let X be a Banach space and N be a norming separable subspace of X^*. Then, every bounded sequence in $L^1(\mu, X)$ has a subsequence whose arithmetic means are $\sigma(X, N)$–Cauchy almost surely.*

Proof. Let (y_m^*) be a countable norm–dense subset of N and (f_n) be a bounded sequence in $L^1(\mu, X)$. Based on succesive applications of Komlós's Theorem (Theorem B) to the $L^1(\mu)$–sequences $(\|f_n\|)_n$, $(\langle y_m^*, f_n\rangle)_n$, $m = 1, 2, ...$, combined with a standard diagonal procedure, we obtain a subset $E \in \Sigma$ of zero–measure, a sequence $(\varphi_n)_{n\geq 0}$ in $L^1(\mu)$ and a subsequence $(f_{n_i})_i$ of (f_n) such that the following two conditions are satisfied

$$\text{(C1)} \quad \frac{1}{k}\sum_{i=1}^{k}\|f_{n_i}(\omega)\| \to \varphi_0(\omega),\ \text{where } \omega \in \Omega \setminus E,$$

$$\text{(C2)} \quad \frac{1}{k}\sum_{i=1}^{k}\langle y_m^*, f_{n_i}(\omega)\rangle \to \varphi_m(\omega),\ \text{where } \omega \in \Omega \setminus E,\ m = 1, 2, ...$$

Let us define $s_k = \frac{1}{k}\sum_{i=1}^{k} f_{n_i} \in L^1(\mu, X)$, for $k \in \mathbb{N}$. According to (C1), $(s_k(\omega))_k \subset X$ is a bounded sequence in X for all $\omega \in \Omega \setminus E$. For the rest of the proof, we assume that $\omega \in \Omega \setminus E$.

On the other hand, since N is a norming subspace for X, the linear continuous mapping

$$T : X \to N^*, x \mapsto T_x, \quad \langle T_x, y^* \rangle = \langle x, y^* \rangle,$$

where $y^* \in N$, is an isometry. Therefore, we may consider (with the above identification) that $(s_k(\omega))_k$ is a bounded sequence in N^*. At this point, we note that if we show that $(s_k(\omega))$ is $\sigma(N^*, N)$–Cauchy, the lemma will be proved. Take two arbitrary increasing subsequences of natural numbers (n_k) and (m_k). Of course, we have to show that $\sigma(N^*, N)$–$\lim_k(s_{n_k}(\omega) - s_{m_k}(\omega)) = 0$.

Suppose on the contrary, that this limit is not zero. Then, we can find a subsequence $(R_k) \subset (s_{n_k}(\omega) - s_{m_k}(\omega))_k$ without any further $\sigma(N^*, N)$–null subsequence. We note that (R_k) is still a bounded sequence in N^*. Since N is separable, N^* is $\sigma(N^*, N)$–sequentially compact. Hence, (R_k) has a subsequence, which we still denote by (R_k), which is $\sigma(N^*, N)$–convergent to some $y^{**} \in N^*$. Bearing in mind (C2), we see that $\lim_k \langle R_k, y_m^* \rangle = 0$, for all $m \in \mathbb{N}$. Moreover, $\{y_m^* : m \in \mathbb{N}\}$ is a total subset for N^*, so we necessarily have that $y^{**} = 0$, and this is a contradiction. □

Theorem 5. (1) *Assume that X^* has the Radon–Nikodým property. Then, every bounded sequence in $L^1(\mu, X)$ has a subsequence whose arithmetic means are weakly Cauchy almost surely.*

(2) *Assume that X is separable. Then, every bounded sequence in $L^1(\mu, X^*)$ has a subsequence whose arithmetic means are weak* convergent almost surely.*

Proof. (1) Suppose that (f_n) is a bounded sequence in $L^1(\mu, X)$. Since f_n are strongly measurable, we obtain null subsets $E_n \in \Sigma$ such that $f_n(\Omega \setminus E_n)$ are separable subsets of X, for all $n \in \mathbb{N}$. Let Z be the closed linear subspace of X which is generated by the union of all subsets $f_n(\Omega \setminus E_n)$. It is clear that Z is separable, and we may assume that $f_n \in L^1(\mu, Z)$. Since X^* has the Radon–Nikodým property and Z is a separable subspace of X, we have that Z^* is also separable [13]. Since it is obvious that Z^* is a norming subspace for Z, we can apply Lemma 3. Hence, (f_n) has a subsequence whose arithmetic means are $\sigma(Z, Z^*)$–Cauchy almost surely and, therefore, $\sigma(X, X^*)$–Cauchy almost surely.

(2) It is clear that the canonical image of X in its bidual is a separable norming subspace for X^*. Suppose that (f_n) is a bounded sequence in $L^1(\mu, X^*)$. Then, we can apply Lemma 3 to the dual pair (X^*, X) to deduce that (f_n) has a subsequence whose arithmetic means are $\sigma(X^*, X)$–Cauchy almost surely. Finally, Alaoglu's Theorem [4, p. 13] allows us to say that the aritmethic means of that subsequence are $\sigma(X^*, X)$–convergent almost surely. □

Remark. Statement (1) of Theorem 5 does not hold for the Banach space ℓ_1. The sequence $f_n = e_n \chi_\Omega(\cdot)$ is clearly $L^1(\mu, \ell_1)$–bounded. If we could apply Theorem 5(1), we would obtain a subsequence (e_{n_k}) such that $(1/k \sum_{i=1}^k e_{n_i})_k$ would be $\sigma(\ell_1, \ell_\infty)$–Cauchy. In particular, we would deduce that the Cesàro method sums every bounded sequence of real numbers, and this is known to be false [12].

Since the class of Banach spaces not containing ℓ_1 and the class of Banach spaces whose duals have the Radon–Nikodým property are quite close to each other (in fact, they coincide in the framework of Banach lattices [6, p. 95]), we see that there is little chance to extend Komlós's Theorem stated as in (1).

The natural question about the possibility of a vector–valued extension of Komlós's Theorem that can be applied to spaces containing ℓ_1 is the background of statement (2) of Theorem 5. We note that, for example, ℓ_1 or ℓ_∞ have separable preduals.

Theorem 6. *Assume that X^* has the Radon–Nikodým property, and let A be a bounded subset in $L^1(\mu, X)$. Then, A is weakly relatively compact if and only if A is uniformly integrable and every sequence in A has a subsequence whose arithmetic means are weakly convergent almost surely.*

Proof. ($\Leftarrow$) It follows directly from Theorem 3. ($\Rightarrow$) On the one hand, A is uniformly integrable by Theorem 3. On the other hand, take $(f_n) \subset A$. Since A is weakly relatively compact, we can find a subsequence (f_{n_k}) which is weakly $L^1(\mu, X)$–convergent to some function f. Furthermore, X^* has the Radon–Nikodým property, thus we may appeal to Theorem 5(1) to obtain a subsequence of (f_{n_k}), that we still denote (f_{n_k}), such that their arithmetic means are $\sigma(X, X^*)$–Cauchy almost surely. From Lemma 2, we deduce that their arithmetic means are $\sigma(X, X^*)$–convergent to f almost surely. □

This theorem allows us to give a different proof of the following well–known fact.

Corollary 6. *Assume that X is reflexive, and let A be a bounded subset in $L^1(\mu, X)$. Then, A is weakly relatively compact if and only if A is uniformly integrable.*

Proof. We have that X^* has the Radon–Nikodým property and X is weakly sequentially complete. Then, the result follows from a combined effort of Theorems 5(1) and 6. □

Perhaps the examples that show to what extent the pointwise ranges of weakly relatively compacts subsets of $L^1(\mu, X)$ can be pathological are the following subsets $\{r_n x_n : n \in \mathbb{N}\}$, where (r_n) denotes the Rademacher functions over $[0, 1]$ and (x_n) is an arbitrary bounded sequence in a Banach space X whose dual has the Radon–Nikodým property [6, p. 117]. Using Theorem 6, we see that the reason of that behaviour is a special type of Banach–Saks property of those sequences (x_n).

Theorem 7. *Assume that X^* has the Radon–Nikodým property. Then, every bounded sequence (x_n) in X has a subsequence (x_{n_k}) such that*

$$\textit{weak–}\lim_{m\to\infty} \frac{\varepsilon_1 x_{n_1} + \varepsilon_2 x_{n_2} + \cdots + \varepsilon_m x_{n_m}}{m} = 0,$$

for some pointwise choices (ε_m) of signs $+1$ or -1.

Proof. According to [6, p. 117], we know that $(r_n x_n)$ tends to zero in the weak topology of $L^1([0, 1], X)$. Therefore, $\{r_n x_n : n \in \mathbb{N}\}$ is a weakly relatively compact subset in $L^1([0, 1], X)$. Then, by Theorem 6, we can find a subsequence of $(r_n x_n)$ whose arithmetic means are weakly convergent almost surely. Moreover, by Lemma 2, we deduce that those arithmetic means are weakly convergent to zero almost surely. □

Theorem 8. *Assume that X^* has the Radon–Nikodým property, and let A be a bounded subset in $L^1(\mu, X)$. Then, A is weakly conditionally compact if and only if A is uniformly integrable.*

Proof. ($\Rightarrow$) It follows from Theorem 4.
($\Leftarrow$) It follows from Theorems 4 and 5(1). □

This theorem was also proved, with entirely different techniques, by Bourgain [2] and Pisier [11] for Banach spaces not containing ℓ_1.

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