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ON NONLINEAR n-WIDTHS

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ABSTRACT. For characterization of best nonlinear approximation, DeVore, Howard, and Micchelli have recently suggested the nonlinear n-width $\delta_n(W, X)$ of a subset W in a normed linear space X. We proved by a topological method that for $\delta_n(W, X)$ and the well-known Aleksandrov n-width $a_n(W, X)$ in a Banach space X the following inequalities hold: $\delta_{2n+1}(W, X) \leq a_n(W, X) \leq \delta_n(W, X)$. Let $K^\alpha_{p,\theta}$ be the unit ball of Besov space $B^\alpha_{p,\theta}$, $\alpha > 0$, $1 \leq p, \theta \leq \infty$, of multivariate periodic functions. Then for approximation in L_q, $1 \leq q \leq \infty$, with some restriction on p, q and α, we established the asymptotic degree of these n-widths: $a_n(K^\alpha_{p,\theta}, L_q) \asymp \delta_n(K^\alpha_{p,\theta}, L_q) \asymp n^{-\alpha}$.

1. INTRODUCTION

In nonlinear approximation such as the rational approximation and the approximation by splines with free knots etc., a function f is approximated by the elements of a nonlinear manifold parametrized continuously by $\mathbf{R}^n$. To characterize the best method of such nonlinear approximations, DeVore, Howard and Micchelli [5] recently have suggested the nonlinear n-width of a subset W in a normed linear space X as

$$\delta_n(W, X) := \inf_{F, M_n} \sup_{x \in W} \|x - M_n(F(x))\|_X$$

with the infimum taken over all continuous mappings F from W into $\mathbf{R}^n$ and all continuous mappings M_n from $\mathbf{R}^n$ into X. (Here instead of d_n we use the symbol δ_n to avoid confusion with the well-known Kolmogorov n-width.) Another similar definition of nonlinear n-width has been introduced by Mathé [10]. For approximation in $L_q(\mathbf{I})$, $1 \leq q \leq \infty$, $\mathbf{I} = [0, 1]$, the unit balls U^r_p, $1 \leq p \leq q \leq \infty$, $r = 1, 2, ...$, of Sobolev spaces $W^r_p(\mathbf{I})$ have nonlinear n-width ([4], [5])

$$\delta_n(U^r_p, L_q(\mathbf{I})) \asymp n^{-r},$$

and the unit balls K^α_p of Besov spaces $B^\alpha_p(\mathbf{I})$, $\alpha > 0$, $p = (\alpha + 1/q)^{-1}$, have nonlinear n-width ([4], [5], [6])

$$\delta_n(K^\alpha_p, L_q(\mathbf{I})) \asymp n^{-\alpha}.$$

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On the other hand, in 1933 Aleksandrov [1] introduced another nonlinear n-width:

$$a_n(W,X) := \inf_{F,C_n} \sup_{x\in W} \|x - F(x)\|_X$$

with the infimum taken over all compact sets $C_n \subset X$ of topological dimensions $\le n$ and all continuous mappings from W into C_n. (In the definition one can be restricted with taking the infimum over all polyhedrons.) For approximation in $L_q(\mathbf{I}^d)$, $1 \le q \le \infty$, the unit balls U_p^α of Sobolev spaces $W_p^\alpha(\mathbf{I}^d)$, $\alpha > (d/p - d/q)_+$, have Aleksandrov n-width

$$a_n(U_p^\alpha, L_q(\mathbf{I}^d)) \asymp n^{-\alpha/d}.$$

This degree was obtained in [12] for $d = 1$ and α natural and recently in [9] for $d > 1$ and $\alpha > (d/p - d/q)_+$, where $x_+ := \max\{0, x\}$ for $x \in \mathbf{R}$.

In this paper we investigate relationships between δ_n and a_n and the degree of these quantities in the space $L_q := L_q(\mathbf{T}^d)$, for the unit balls $K_{p,\theta}^\alpha$ in Besov spaces $B_{p,\theta}^\alpha(\mathbf{T}^d)$, $\alpha > 0, 1 \le p \le \infty$, of multivariate periodic functions, where $\mathbf{T}^d = [-\pi, \pi]^d$ denotes the d-dimensional torus.

The main results of this note read as follows.

(i) For any compact set W in the Banach space X we have

$$\delta_{2n+1}(W,X) \le a_n(W,X), \tag{1.1}$$

$$a_n(W,X) \le \delta_n(W,X). \tag{1.2}$$

(ii) If $1 \le p \le q \le \infty$, $\alpha > d/p - d/q$ or $1 < q < p < \infty$, $\alpha > 0$, then

$$a_n(K_{p,\theta}^\alpha, L_q(\mathbf{T}^d)) \asymp \delta_n(K_{p,\theta}^\alpha, L_q(\mathbf{T}^d)) \asymp n^{-\alpha/d}. \tag{1.3}$$

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The assertion (i) shows that a_n and δ_n are closely related. However, we will give an example stating that they are strictly different. An inequality similar to (1.1) was proved by Mathé, for the nonlinear n-width introduced in [10]. From (1.1-2) it follows that for $1 \le p, q \le \infty$ and $\alpha > (d/p - d/q)_+$:

$$\delta_n(U_p^\alpha, L_q(\mathbf{I}^d)) \asymp n^{-\alpha/d}.$$

2. Comparison of a_n and δ_n

First we prove (1.1) and (1.2). For this purpose we need some results from topology. The dimension of a topological space X is the maximal nonnegative integer n such that for any finite open covering Ω for X there exists a finite open covering Ω' of multiplicity $\le n + 1$ for X inserted into Ω. (Recall that Ω' is said to be inserted into Ω, if every element of Ω' is a subset of at least one element of Ω). Denote by $C(X,Y)$ the set of all continuous mappings from the topological space X into the topological space Y.

Lemma 2.1 (Pontryagin-Nöbeling, cf. [2]). *Every n-dimensional compactum is homeomorphic to a subset of $\mathbf{R}^{2n+1}$.*

Lemma 2.2. *Let T be an n-dimensional metric space. Assume that there exists $F \in C(T,X)$ such that $F(T)$ is a compact subset in X. Then for arbitrary $\varepsilon > 0$ there exist an n-dimensional polyhedron $P \subset X$ and $F_\varepsilon \in C(T,P)$ such that*

$$\sup_{x\in T} \|F(x) - F_\varepsilon(x)\|_X < \varepsilon. \tag{2.1}$$

Proof. The lemma is a slight generalization of Aleksandrov's theorem stating it for the case when $X = \mathbf{R}^m \quad (m \geq n), \quad F(T) \subset [0,1]^m$ (cf. [2]). Obviously, Aleksandrov's theorem can be extended to the case when X is finite dimensional and $F(X)$ is included in the unit ball in X. We now prove the lemma in the general case. Given arbitrary $\varepsilon > 0$, we find an m-dimensional linear manifold $L \subset X \quad (m > n)$ and $G \in C(F(T), L)$ such that (cf., e.g., [13])

$$\sup_{y \in F(T)} \|G(y) - G_\varepsilon(y)\|_X < \varepsilon/2. \tag{2.2}$$

We have $G := G_\varepsilon F \in C(T, L)$. Thus, there exist an n-dimensional polyhedron $P \subset L$ and $F_\varepsilon \in C(T, P)$ such that

$$\sup_{x \in T} \|G(x) - F_\varepsilon(x)\|_X < \varepsilon/2.$$

This and (2.2) give (2.1). The lemma is proved.

We are now in a position to prove the following

Theorem 2.1. *For any compact subset W in the Banach space X the inequalities* (1.1) - (1.2) *hold.*

Proof. Given arbitrary $\varepsilon > 0$, by the definition of a_n there exist an n-dimensional compact set $K \subset X$ and $F \in C(W, K)$ such that

$$\sup_{x \in W} \|x - F(x)\|_X < a + \varepsilon,$$

where $a := a_n(W, X)$. There exists by Lemma 2.1 a homeomorphism G from K onto a compact subset of $\mathbf{R}^{2n+1}$. A theorem of Dowker [8] states that every continuous mapping from a compact subset of $\mathbf{R}^n$ into the Banach space X can be extended to a continuous mapping from the whole $\mathbf{R}^m$ into X. Let $M_{2n+1} : \quad \mathbf{R}^{2n+1} \longrightarrow X$ be such an extension of the mapping G^{-1} inverse to G. We have $H := GF \in C(W, \mathbf{R}^{2n+1})$ and therefore,

$$\delta_{2n+1}(W, X) \leq \sup_{x \in W} \|x - M_{2n+1}(H(x))\|_X = \sup_{x \in W} \|x - F(x)\|_X < a + \varepsilon.$$

This proves (1.1).

We next prove (1.2). Given arbitrary $\varepsilon > 0$, by the definition of δ_n there exist $H \in C(W, \mathbf{R}^n)$ and $M_n \in C(\mathbf{R}^n, X)$ such that

$$\sup_{x \in W} \|x - M_n(H(x))\|_X < \delta + \varepsilon/2, \tag{2.3}$$

where $\delta := \delta_n(W, X)$. The set $T := H(W) \subset \mathbf{R}^n$ is compact as well as $M_n(T)$. Thus by Lemma 2.2 there exist an n-dimensional polyhedron $P \subset X$ and $F_\varepsilon \in C(T, P)$ such that

$$\sup_{y \in T} \|M_n(y) - F_\varepsilon(y)\|_X < \varepsilon/2.$$

Hence, for $F := F_\varepsilon H \in C(W, P)$ we have by (2.3)

$$\sup_{x \in W} \|x - F(x)\|_X \leq \sup_{x \in W} \|x - M_n(H(x))\|_X$$

$$+ \sup_{x \in W} \|M(H(x)) - F_\varepsilon(H(x))\|_X \leq \delta + \varepsilon.$$

This proves (1.2). The theorem is proved.

From Theorem 2.1 one can see that in many cases of W and X $a_n(W,X)$ and $\delta_n(W,X)$ have the same degree. However, in general, these quantities are strictly different approximation characteristics as the following example shows. Let $S^n := \{x \in \mathbf{R}^m : \|x\| = 1, \quad x_j = 0, \quad n+2 \le j \le m\} \quad (n+2 \le m)$, then clearly, $a_n(S^n, \mathbf{R}^m) = 0$. On the other hand, by using Borsuk's antipodality theorem, for any $F \in C(S^n, \mathbf{R}^n)$ there exists $y \in S^n$ such that $F(y) = F(-y)$. Thus for any $M_n \in C(\mathbf{R}^n, \mathbf{R}^m)$

$$\sup_{x \in S^n} \|x - M_n(F(x))\| \ge \max\{\|y + M_n(F(y))\|, \|y - M_n(F(y))\|\} \ge \|y\|_{\mathbf{R}^m}.$$

This means that $\delta_n(S^n, \mathbf{R}^m) \ge 1$.

The Bernstein n-width of a symmetric subset W in the normed linear space X is defined by

$$b_n(W,X) := \sup_{L_{n+1}} \sup\{b : \quad bBX \cap L_{n+1} \subset W\}$$

with the outer supremum taken over all $(n+1)$-dimensional linear subspaces of X. This quantity introduced by Tikhomirov (cf. [13]), is quite suitable for lower estimating various n-widths. In our case we have

$$2a_n(W,X) \ge b_n(W,X), \tag{2.4}$$

$$\delta_n(W,X) \ge b_n(W,X).$$

The first inequality was proved by Tikhomirov (cf. [13]) and the second one by DeVore, Howard and Micchelli [5]. We temporarily use the symbols γ_n for either a_n or δ_n.

Lemma 2.3. *If Y is a subspace of the linear normed space X, $W \subset Y$ and $P : X \longrightarrow Y$ is a continuous linear projection, then*

$$\gamma_n(W,X) \ge \|P\|^{-1}\gamma_n(W,Y).$$

Proof. The lemma in both cases can be proved by a similar argument. We will prove it, for example, for δ_n. Putting $F' := F$ and $M' := PM_n \in C(\mathbf{R}^n, Y)$ for any $F \in C(W, \mathbf{R}^n)$, $M \in C(\mathbf{R}^n, X)$ and $x \in W$, we have for $x \in W$

$$\begin{aligned}\|P\|\,\|x - M_n(F(x))\| &\ge \|P(x) - PM_n(F(x))\|_X \\ &= \|x - M(F'(x))\|_X.\end{aligned}$$

Hence,

$$\|P\|\delta_n(W,X) \ge \inf_{F', M'_n} \sup_{x \in W} \|x - M'_n(F'(x)))\|_X.$$

This proves the lemma for δ_n because the right side of the last inequality is not less than $\delta_n(W,Y)$.

Lemma 2.4. *Let the linear space L be normed by $\|.\|_X$ and $\|.\|_Y$ and $W \subset L$. Assume that W is compact in the norm $\|.\|_X$ and $\gamma_n(W,X) > 0$ for some nonnegative integer n. Then for every nonnegative integer s*

$$\gamma_{n+s}(W,Y) \le \gamma_n(W,X)\gamma_s(BX,Y),$$

where $BX = \{x \in L : \quad \|x\|_X \le 1\}$.

Proof. This lemma has been proved by Khodulev (oral communication, 1986) for a_n. We will prove it for δ_n. For any $F \in C(W, \mathbf{R}^n)$, $M_n \in C(\mathbf{R}^n, X)$ and $H \in C(BX, \mathbf{R}^s)$, $M_s \in C(\mathbf{R}^s, Y)$ put $G(x) := x - M_n F(x)$ and

$$\lambda := \sup_{x \in W} \|G(x)\|_X.$$

Since W is compact in the norm $\|.\|_X$ and $\delta_n(W, X) > 0$, we have $0 < \lambda < \infty$. We define $F^* \in C(W, \mathbf{R}^{n+s})$ and $M^*_{n+s} \in C(\mathbf{R}^n, Y)$ by

$$F^*(x) := (F(x), H(G(x)/\lambda)), \quad x \in W,$$

and

$$M^*_{n+s}(\xi, \eta) := M_n(\xi) + \lambda M_s(\eta), \quad \xi \in \mathbf{R}^n, \eta \in \mathbf{R}^s.$$

From the equality

$$x - M^*_{n+s} F^*(x) = \lambda\{G(x)/\lambda - M_s H(G(x))/\lambda)\}$$

and the inclusion $G(x)/\lambda \in BX$, we obtain

$$\sup_{x \in W} \|x - M^*_{n+s} F^*(x)\|_Y \leq \sup_{x \in W} \|x - M_n F(x)\|_X \sup_{x \in BX} \|x - M_s H(x)\|_Y.$$

This proves the lemma for δ_n.

For $1 \leq p \leq \infty$, $\mathbf{R}^m$ can be equipped by the norm

$$\|x\|_{l^m_p} = (\sum_{k=1}^m |x_k|^p)^{1/p} \quad (p < \infty)$$

with the usual change to the max norm when $p = \infty$. Let B^m_p denote the unit ball in l^m_p.

Lemma 2.5. *If $1 \leq p, q \leq \infty$, $m > n$, then*

$$a_n(B^m_p, l^m_q) = \begin{cases} (n+1)^{1/q - 1/p}, & \text{for} \quad p < q, \\ 1, & \text{for} \quad p = q, \\ (m-n)^{1/q-1/p}, & \text{for} \quad p > q. \end{cases}$$

Proof. This lemma was proved by Tikhomirov [13] for $p = q$, by Stesin [12] for $p < q$ and by Khodulev for $p > q$. In the last case the inequality $a_n(B^m_p, l^m_q) \leq (m-n)^{1/q-1/p}$ is easily verified and the inequality $a_n(B^m_p, l^m_q) \geq (m-n)^{1/q-1/p}$ immediately follows from the first case and Lemma 2.4 by setting $X := l^m_p$, $Y := l^m_q$, $W := B^m_p$ and $s := m - n - 1$.

3. Nonlinear widths in Besov spaces

If $\alpha > 0$ and $1 \leq p, \theta \leq \infty$, then the Besov space $B^\alpha_{p,\theta} = B^\alpha_{p,\theta}(\mathbf{T}^d)$ consists of all functions f defined on $\mathbf{T}^d$ such that the norm

$$\|f\|_{B^\alpha_{p,\theta}} := \|f\|_p + |f|_{B^\alpha_{p,\theta}}$$

is finite, where

$$|f|_{B^\alpha_{p,\theta}} := \begin{cases} (\int_0^\infty (t^{-\alpha}\omega_r(f,t)_p)^\theta dt/t)^{1/\theta}, & 1 < \theta < \infty, \\ \sup_{t \geq 0} t^{-\alpha}\omega_r(f,t)_p), & \theta = \infty, \end{cases}$$

$r > \alpha$, $\omega_r(f, .)_p$ is the modulus of smoothness of order r of the function $f \in L_p(\mathbf{T}^d)$. It is well known that different values of $r > \alpha$ give equivalent norms (see, e.g., [11]). Let

$$V_m(t) = 1 + 2\sum_{k=1}^{m} \cos kt + 2\sum_{k=m+1}^{2m} ((2m-k)/m)\cos kt$$

be the de la Vallée Poussin kernel. Then the multidimensional de la Vallée Poussin sums are defined by

$$V_m(x) := \prod_{j=1}^{d} V_m(x_j)$$

for $m \in \mathbf{N}$. For functions f on $\mathbf{T}^d$ consider the convolution operator $V_m f := f * V_m$ defining the de la Vallée Poussin sum of f. The differences of successive de la Vallée Poussin sums are defined by

$$\Phi_0 f := V_1 f, \quad \Phi_k := V_{2^k} f - V_{2^{k-1}} f, \quad k = 1, 2, \dots .$$

If $\alpha > 0,\ 1 \le p, \theta \le \infty$, then (cf. [11])

$$\|f\|_{B^\alpha_{p,\theta}} \asymp \begin{cases} (\sum_{k=0}^{\infty} (2^{\alpha k}\|\Phi_k f\|_p)^\theta)^{1/\theta}, & 1 < \theta < \infty, \\ \sup_k 2^{\alpha k}\|\Phi_k f\|_p, & \theta = \infty. \end{cases} \tag{3.1}$$

Let T_m denote the set of trigonometric polynomials of order $\le m$ in each variable x_j, $j = 1, 2, ..., d$.

If $1 \le p \le \infty$, then for every $f \in T_m$

$$\|f\|_p \asymp m^{-d/p}(\sum_{k \in P_m} |f(hk)|^p)^{1/p} \tag{3.2}$$

with the change to max norm when $p = \infty$, where $P_m := \{k \in \mathbf{Z}^d : -2m \le k_j < 2m, j = 1, ..., d\}, h = \pi/2m$ (cf. [14]). Let

$$K^\alpha_{p,\theta} := \{f \in B^\alpha_{p,\theta} : \quad \|f\|_{B^\alpha_{p,\theta}} \le 1\}$$

be the unit ball in $B^\alpha_{p,\theta}$. In what follows we again use the symbol γ_n for either a_n or δ_n.

Lemma 3.1. *If $\alpha > 0,\ 1 \le p, \theta, q \le \infty$ and $\{n_k\}_{k=0}^\infty$ is a sequence of nonnegative integers such that $\sum_{k=0}^\infty n_k \le n$, then for each nonnegative integer s*

$$\gamma_n(K^\alpha_{p,\theta}, L_q) \ll 2^{(d/p-d/q)s}\gamma_{n_k}(B_p^{d(s+2)}, l_q^{d(s+2)})$$

$$+ \sum_{k=s+1}^{\infty} 2^{-(\alpha-d/p+d/q)k}\gamma_{n_k}(B_p^{d(s+2)}, l_q^{d(s+2)}).$$

Proof. The lemma can be proved for both linear and Aleksandrov n-widths in similar ways. We will prove it, for example, for Aleksandrov n-widths. Moreover, because of the inclusion $B_{p,\theta'} \subset B_{p,\theta}, \theta' < \theta$, it is enough to prove the lemma for $\theta = \infty$.

If $f \in K^\alpha_{p,\theta}$, then

$$f = V_{2^s} f + \sum_{k=s+1}^{\infty} \Phi_k f,$$

the series converging in L_p, and

$$\|V_{2^s} f\|_p \le c \quad (c > 0), \tag{3.3}$$

$$\|\Phi_k f\|_p \le 2^{-\alpha}. \tag{3.4}$$

The inequality (3.3) follows from the inequalities $\|V_m f\|_p \le \|V_m\|_1 \|f\|_p$ and $\|V_m\|_1 \le c' \quad (c' > 0)$ for any m, the inequality (3.4) follows from (3.1). Note that the operators V_{2^s} and $\Phi_k, k > s$, are continuous in L_p and

$$V_{2^s} f \in B_s := \{g \in T_{2^{s+1}} : \|g\|_p \le c\}$$

and

$$\Phi_k f \in B_k := \{g \in T_{2^{k+1}} : \|g\|_p \le 2^{-\alpha k}\}, \quad k = s+1, s+2, \dots .$$

Let $F_k : B_k \longrightarrow C_k$ be a continuous mapping from B_k into $C_k \subset T_{2^{k+1}}, k = s, s+1, ...,$ where C_k are polyhedrons of dimension n_k. We define a polyhedron $C := \sum_{k=s}^{\infty} C_k$ of dimension $\le n$ and the continuous mapping $F : K^{\alpha}_{p,\infty} \longrightarrow C$ by

$$F(f) := F_s(V_{2^s} f) + \sum_{k=s}^{\infty} F_k(\Phi_k f).$$

We have for $f \in K^{\alpha}_{p,\infty}$

$$\|f - F(f)\|_q \le \|V_{2^s} f - F_s(V_{2^s} f)\|_q + \sum_{k=s+1}^{\infty} \|\Phi_k f - F_k(\Phi_k f)\|_q.$$

Hence we get

$$a_n(K^{\alpha}_{p,\infty}, L_q) \le a_{n_s}(B_s, T_{2^{s+1}} \cap L_q) + \sum_{k=s+1}^{\infty} a_{n_k}(B_k, T_{2^{k+1}} \cap L_q). \tag{3.5}$$

By using (3.2) we can see

$$a_{n_s}(B_s, T_{2^{s+1}} \cap L_q) \approx 2^{(d/p - d/q)s} a_{n_s}(B_p^{d(k+s)}, l_q^{d(s+2)}),$$

$$a_{n_k}(B_k, T_{2^{k+1}} \cap L_q) \approx 2^{(\alpha - d/p + d/q)k} a_{n_k}(B_p^{d(k+2)}, l_q^{d(k+2)})$$

for $k > s$. This and (3.5) prove the lemma.

Theorem 3.1. *If $1 \le p, \theta, q \le \infty$ and $\alpha > (d/p - d/q)_+$, then*

$$a_n(K^{\alpha}_{p,\theta}, L_q) \ll n^{-\alpha/d}.$$

Proof. First we consider the case $p < q$. For a given natural number n (large enough) let s be the natural number defined by the condition $2^{d(s+3)} \le n < 2^{d(s+4)}$. Taking a fixed number δ with $0 < \delta < (\alpha - d/p + d/q)/(1/p - 1/q)$, we define the sequence $\{n_k\}_{k=s}^{\infty}$ by $n_s = 2^{d(s+2)}$ and $n_k = [cn^{-2\delta d(k-s)}]$ for $k > s$ with constant c depending on δ only and chosen such that $\sum_s^{\infty} n_k \le n$, where $[t]$ denotes the integer part of $t \in \mathbf{R}$. To prove the theorem we now apply Lemma 3.1. Obviously,

$$a_{n_s}(B_p^{d(s+2)}, l_q^{d(s+2)}) = 0$$

and by Lemma 2.5

$$a_{n_k}(B_p^{d(k+2)}, l_q^{d(k+2)}) \approx (n2^{-\delta(k-s)})^{d/q - d/p}.$$

Hence by Lemma 3.1 we obtain

$$a_n(K^\alpha_{p,\theta}, L_q) \ll \sum_{k=s+1}^{\infty} 2^{-(\alpha-d/p+d/q)k}(n2^{-\delta(k-s)})^{(d/q-d/p)}$$

$$= n^{d/q-d/p}2^{\delta s(d/q-d/p)} \sum_{k=s+1}^{\infty} 2^{-\beta k},$$

where $\beta = \alpha - d/p + d/q - \delta(d/p - d/q) > 0$. Therefore,

$$a_n(K^\alpha_{p,\theta}, L_q) \ll n^{d/q-d/p}2^{\delta s(d/q-d/p)}2^{-\beta s} \asymp n^{-\alpha/d}.$$

Thus, the theorem is proved in the case $p < q$. In the case $p \geq q$ it can be proved in a similar way by setting $n_s = n$ and $n_k = 0$ for $k > s$.

Lemma 3.2. *If* $1 \leq p \leq q \leq \infty, \quad 1 \leq \theta \leq \infty$ *and* $\alpha > 0$*, then for any* $f \in T_m$

$$\|f\|_{B^\alpha_{p,\theta}} \ll m^\alpha \|f\|_q.$$

Proof. Since $\|f\|_p \leq (2\pi)^{d/p-d/q}\|f\|_q$ for $f \in L_q(\mathbf{T}^d)$ and $1 \leq p < q \leq \infty$, it is enough to prove the lemma for the case $p = q$. For a given natural number m (large enough), let s be the natural number such that $2^{s-1} \leq m < 2^s$. If $f \in T_m$; then $f \in T_{2^s}$ and

$$f = \sum_{k=0}^{s} \Phi_k f.$$

Hence, (3.1) and the inequality $\|\Phi_k f\|_p \ll \|f\|_p, k = 0, 1, 2, ...,$ give

$$\|f\|_{B^\alpha_{p,\theta}} \leq \sum_{k=0}^{s} 2^{\alpha k}\|\Phi_k f\|_p \ll \|f\|_p \sum_{k=0}^{s} 2^{\alpha k} \ll m^\alpha \|f\|_p.$$

Theorem 3.2. *If* $1 \leq p \leq q \leq \infty$ *or* $1 < q < p < \infty, \quad 1 \leq \theta \leq \infty$ *and* $\alpha > 0$*, then*

$$a_n(K^\alpha_{p,\theta}, L_q) \gg n^{-\alpha/d}.$$

Proof. If $1 \leq p \leq q \leq \infty$, then from Lemma 3.2 it follows that $b_n(K^\alpha_{p,\theta}, L_q) \gg n^{-\alpha/d}$. This and inequality (2.4) prove the theorem in the case $1 \leq p \leq q \leq \infty$. Let m be the natural number defined by the condition $(2m-1)^d < 2n \leq (2m+1)^d$. By Lemma 3.2 we have $K^\alpha_{p,\theta} \supset B := \{f \in T_m : \|f\|_p \leq cm^{-\alpha}\}$ with the constant c and, therefore

$$a_n(K^\alpha_{p,\theta}, L_q) \geq a_n(B, L_q). \tag{3.6}$$

For a function f defined on $\mathbf{T}^d$, let

$$(S_m f)(x) := \sum_{k \in \triangle_m} f_k e^{i\langle k,x\rangle},$$

where f_k is the k-th Fourier coefficient of f, $\langle k, x\rangle$ is the scalar product of k and x and $\triangle_m = \{k \in \mathbf{Z}^d : |k_j| \leq m, \quad j = 1, 2, ..., d\}$. Then S_m is a linear continuous projection from L_q onto T_m and $\|S_m\|_{L_q \to L_q} \leq c' \quad (1 < q < \infty)$ for some constant c' independent of m. Applying Lemma 2.3, we get

$$a_n(B, L_q) \gg a_n(B, L_q \cap T_m). \tag{3.7}$$

By use of the Marcinkiewicz theorem (see [14] for univariate polynomials; the theorem and its proof for multivariate polynomials are completely analogous) one can verify

$$a_n(B, L_q \cap T_m) \gg m^{-\alpha} m^{(1/p-1/q)d} a_n(B_p^N, l_q^N),$$

where $N = (2m+1)^d$. Combining this and (3.6)-(3.7), by Lemma 2.5 we prove the theorem in the case $1 < q < p < \infty$.

From Theorems 3.1-2 and 2.1 we obtain (1.3).

Remark . After submitting this paper to Proceedings of the AMS we received a manuscript by DeVore, Kyriazis, Leviatan and Tikhomirov [7] which gives some interesting results close to our paper. In particular, the authors obtained asymptotic degree of nonlinear n-widths of $K^\alpha_{p,\theta}$ in spaces L_q with $0 < q \le \infty$, using wavelet decompositions.

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