

ON NONLINEAR n -WIDTHS

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ABSTRACT. For characterization of best nonlinear approximation, DeVore, Howard, and Micchelli have recently suggested the nonlinear n -width $\delta_n(W, X)$ of a subset W in a normed linear space X . We proved by a topological method that for $\delta_n(W, X)$ and the well-known Aleksandrov n -width $a_n(W, X)$ in a Banach space X the following inequalities hold: $\delta_{2n+1}(W, X) \leq a_n(W, X) \leq \delta_n(W, X)$. Let $K_{p,\theta}^\alpha$ be the unit ball of Besov space $B_{p,\theta}^\alpha$, $\alpha > 0$, $1 \leq p, \theta \leq \infty$, of multivariate periodic functions. Then for approximation in L_q , $1 \leq q \leq \infty$, with some restriction on p, q and α , we established the asymptotic degree of these n -widths: $a_n(K_{p,\theta}^\alpha, L_q) \approx \delta_n(K_{p,\theta}^\alpha, L_q) \approx n^{-\alpha}$.

1. INTRODUCTION

In nonlinear approximation such as the rational approximation and the approximation by splines with free knots etc., a function f is approximated by the elements of a nonlinear manifold parametrized continuously by \mathbf{R}^n . To characterize the best method of such nonlinear approximations, DeVore, Howard and Micchelli [5] recently have suggested the nonlinear n -width of a subset W in a normed linear space X as

$$\delta_n(W, X) := \inf_{F, M_n} \sup_{x \in W} \|x - M_n(F(x))\|_X$$

with the infimum taken over all continuous mappings F from W into \mathbf{R}^n and all continuous mappings M_n from \mathbf{R}^n into X . (Here instead of d_n we use the symbol δ_n to avoid confusion with the well-known Kolmogorov n -width.) Another similar definition of nonlinear n -width has been introduced by Mathé [10]. For approximation in $L_q(\mathbf{I})$, $1 \leq q \leq \infty$, $\mathbf{I} = [0, 1]$, the unit balls U_p^r , $1 \leq p \leq q \leq \infty$, $r = 1, 2, \dots$, of Sobolev spaces $W_p^r(\mathbf{I})$ have nonlinear n -width ([4], [5])

$$\delta_n(U_p^r, L_q(\mathbf{I})) \approx n^{-r},$$

and the unit balls K_p^α of Besov spaces $B_p^\alpha(\mathbf{I})$, $\alpha > 0$, $p = (\alpha + 1/q)^{-1}$, have nonlinear n -width ([4], [5], [6])

$$\delta_n(K_p^\alpha, L_q(\mathbf{I})) \approx n^{-\alpha}.$$

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On the other hand, in 1933 Aleksandrov [1] introduced another nonlinear n -width:

$$a_n(W, X) := \inf_{F, C_n} \sup_{x \in W} \|x - F(x)\|_X$$

with the infimum taken over all compact sets $C_n \subset X$ of topological dimensions $\leq n$ and all continuous mappings from W into C_n . (In the definition one can be restricted with taking the infimum over all polyhedrons.) For approximation in $L_q(\mathbf{I}^d)$, $1 \leq q \leq \infty$, the unit balls U_p^α of Sobolev spaces $W_p^\alpha(\mathbf{I}^d)$, $\alpha > (d/p - d/q)_+$, have Aleksandrov n -width

$$a_n(U_p^\alpha, L_q(\mathbf{I}^d)) \approx n^{-\alpha/d}.$$

This degree was obtained in [12] for $d = 1$ and α natural and recently in [9] for $d > 1$ and $\alpha > (d/p - d/q)_+$, where $x_+ := \max\{0, x\}$ for $x \in \mathbf{R}$.

In this paper we investigate relationships between δ_n and a_n and the degree of these quantities in the space $L_q := L_q(\mathbf{T}^d)$, for the unit balls $K_{p,\theta}^\alpha$ in Besov spaces $B_{p,\theta}^\alpha(\mathbf{T}^d)$, $\alpha > 0, 1 \leq p \leq \infty$, of multivariate periodic functions, where $\mathbf{T}^d = [-\pi, \pi]^d$ denotes the d -dimensional torus.

The main results of this note read as follows.

(i) For any compact set W in the Banach space X we have

$$(1.1) \quad \delta_{2n+1}(W, X) \leq a_n(W, X),$$

$$(1.2) \quad a_n(W, X) \leq \delta_n(W, X).$$

(ii) If $1 \leq p \leq q \leq \infty$, $\alpha > d/p - d/q$ or $1 < q < p < \infty$, $\alpha > 0$, then

$$(1.3) \quad a_n(K_{p,\theta}^\alpha, L_q(\mathbf{T}^d)) \approx \delta_n(K_{p,\theta}^\alpha, L_q(\mathbf{T}^d)) \approx n^{-\alpha/d}.$$

The assertion (i) shows that a_n and δ_n are closely related. However, we will give an example stating that they are strictly different. An inequality similar to (1.1) was proved by Mathé, for the nonlinear n -width introduced in [10]. From (1.1-2) it follows that for $1 \leq p, q \leq \infty$ and $\alpha > (d/p - d/q)_+$:

$$\delta_n(U_p^\alpha, L_q(\mathbf{I}^d)) \approx n^{-\alpha/d}.$$

2. COMPARISON OF a_n AND δ_n

First we prove (1.1) and (1.2). For this purpose we need some results from topology. The dimension of a topological space X is the maximal nonnegative integer n such that for any finite open covering Ω for X there exists a finite open covering Ω' of multiplicity $\leq n + 1$ for X inserted into Ω . (Recall that Ω' is said to be inserted into Ω , if every element of Ω' is a subset of at least one element of Ω). Denote by $C(X, Y)$ the set of all continuous mappings from the topological space X into the topological space Y .

Lemma 2.1 (Pontryagin-Nöbeling, cf. [2]). *Every n -dimensional compactum is homeomorphic to a subset of \mathbf{R}^{2n+1} .*

Lemma 2.2. *Let T be an n -dimensional metric space. Assume that there exists $F \in C(T, X)$ such that $F(T)$ is a compact subset in X . Then for arbitrary $\varepsilon > 0$ there exist an n -dimensional polyhedron $P \subset X$ and $F_\varepsilon \in C(T, P)$ such that*

$$(2.1) \quad \sup_{x \in T} \|F(x) - F_\varepsilon(x)\|_X < \varepsilon.$$

Proof. The lemma is a slight generalization of Aleksandrov's theorem stating it for the case when $X = \mathbf{R}^m$ ($m \geq n$), $F(T) \subset [0, 1]^m$ (cf. [2]). Obviously, Aleksandrov's theorem can be extended to the case when X is finite dimensional and $F(X)$ is included in the unit ball in X . We now prove the lemma in the general case. Given arbitrary $\varepsilon > 0$, we find an m -dimensional linear manifold $L \subset X$ ($m > n$) and $G \in C(F(T), L)$ such that (cf., e.g., [13])

$$(2.2) \quad \sup_{y \in F(T)} \|G(y) - G_\varepsilon(y)\|_X < \varepsilon/2.$$

We have $G := G_\varepsilon F \in C(T, L)$. Thus, there exist an n -dimensional polyhedron $P \subset L$ and $F_\varepsilon \in C(T, P)$ such that

$$\sup_{x \in T} \|G(x) - F_\varepsilon(x)\|_X < \varepsilon/2.$$

This and (2.2) give (2.1). The lemma is proved.

We are now in a position to prove the following

Theorem 2.1. *For any compact subset W in the Banach space X the inequalities (1.1) - (1.2) hold.*

Proof. Given arbitrary $\varepsilon > 0$, by the definition of a_n there exist an n -dimensional compact set $K \subset X$ and $F \in C(W, K)$ such that

$$\sup_{x \in W} \|x - F(x)\|_X < a + \varepsilon,$$

where $a := a_n(W, X)$. There exists by Lemma 2.1 a homeomorphism G from K onto a compact subset of \mathbf{R}^{2n+1} . A theorem of Dowker [8] states that every continuous mapping from a compact subset of \mathbf{R}^n into the Banach space X can be extended to a continuous mapping from the whole \mathbf{R}^m into X . Let $M_{2n+1} : \mathbf{R}^{2n+1} \rightarrow X$ be such an extension of the mapping G^{-1} inverse to G . We have $H := GF \in C(W, \mathbf{R}^{2n+1})$ and therefore,

$$\delta_{2n+1}(W, X) \leq \sup_{x \in W} \|x - M_{2n+1}(H(x))\|_X = \sup_{x \in W} \|x - F(x)\|_X < a + \varepsilon.$$

This proves (1.1).

We next prove (1.2). Given arbitrary $\varepsilon > 0$, by the definition of δ_n there exist $H \in C(W, \mathbf{R}^n)$ and $M_n \in C(\mathbf{R}^n, X)$ such that

$$(2.3) \quad \sup_{x \in W} \|x - M_n(H(x))\|_X < \delta + \varepsilon/2,$$

where $\delta := \delta_n(W, X)$. The set $T := H(W) \subset \mathbf{R}^n$ is compact as well as $M_n(T)$. Thus by Lemma 2.2 there exist an n -dimensional polyhedron $P \subset X$ and $F_\varepsilon \in C(T, P)$ such that

$$\sup_{y \in T} \|M_n(y) - F_\varepsilon(y)\|_X < \varepsilon/2.$$

Hence, for $F := F_\varepsilon H \in C(W, P)$ we have by (2.3)

$$\begin{aligned} \sup_{x \in W} \|x - F(x)\|_X &\leq \sup_{x \in W} \|x - M_n(H(x))\|_X \\ &+ \sup_{x \in W} \|M_n(H(x)) - F_\varepsilon(H(x))\|_X \leq \delta + \varepsilon. \end{aligned}$$

This proves (1.2). The theorem is proved.

From Theorem 2.1 one can see that in many cases of W and X $a_n(W, X)$ and $\delta_n(W, X)$ have the same degree. However, in general, these quantities are strictly different approximation characteristics as the following example shows. Let $S^n := \{x \in \mathbf{R}^m : \|x\| = 1, x_j = 0, n + 2 \leq j \leq m\}$ ($n + 2 \leq m$), then clearly, $a_n(S^n, \mathbf{R}^m) = 0$. On the other hand, by using Borsuk's antipodality theorem, for any $F \in C(S^n, \mathbf{R}^n)$ there exists $y \in S^n$ such that $F(y) = F(-y)$. Thus for any $M_n \in C(\mathbf{R}^n, \mathbf{R}^m)$

$$\sup_{x \in S^n} \|x - M_n(F(x))\| \geq \max\{\|y + M_n(F(y))\|, \|y - M_n(F(y))\|\} \geq \|y\|_{\mathbf{R}^m}.$$

This means that $\delta_n(S^n, \mathbf{R}^m) \geq 1$.

The Bernstein n -width of a symmetric subset W in the normed linear space X is defined by

$$b_n(W, X) := \sup_{L_{n+1}} \sup\{b : bBX \cap L_{n+1} \subset W\}$$

with the outer supremum taken over all $(n + 1)$ -dimensional linear subspaces of X . This quantity introduced by Tikhomirov (cf. [13]), is quite suitable for lower estimating various n -widths. In our case we have

$$(2.4) \quad 2a_n(W, X) \geq b_n(W, X),$$

$$\delta_n(W, X) \geq b_n(W, X).$$

The first inequality was proved by Tikhomirov (cf. [13]) and the second one by DeVore, Howard and Micchelli [5]. We temporarily use the symbols γ_n for either a_n or δ_n .

Lemma 2.3. *If Y is a subspace of the linear normed space X , $W \subset Y$ and $P : X \rightarrow Y$ is a continuous linear projection, then*

$$\gamma_n(W, X) \geq \|P\|^{-1} \gamma_n(W, Y).$$

Proof. The lemma in both cases can be proved by a similar argument. We will prove it, for example, for δ_n . Putting $F' := F$ and $M' := PM_n \in C(\mathbf{R}^n, Y)$ for any $F \in C(W, \mathbf{R}^n)$, $M \in C(\mathbf{R}^n, X)$ and $x \in W$, we have for $x \in W$

$$\begin{aligned} \|P\| \|x - M_n(F(x))\| &\geq \|P(x) - PM_n(F(x))\|_X \\ &= \|x - M(F'(x))\|_X. \end{aligned}$$

Hence,

$$\|P\| \delta_n(W, X) \geq \inf_{F', M'_n} \sup_{x \in W} \|x - M'_n(F'(x))\|_X.$$

This proves the lemma for δ_n because the right side of the last inequality is not less than $\delta_n(W, Y)$.

Lemma 2.4. *Let the linear space L be normed by $\|\cdot\|_X$ and $\|\cdot\|_Y$ and $W \subset L$. Assume that W is compact in the norm $\|\cdot\|_X$ and $\gamma_n(W, X) > 0$ for some nonnegative integer n . Then for every nonnegative integer s*

$$\gamma_{n+s}(W, Y) \leq \gamma_n(W, X) \gamma_s(BX, Y),$$

where $BX = \{x \in L : \|x\|_X \leq 1\}$.

Proof. This lemma has been proved by Khodulev (oral communication, 1986) for a_n . We will prove it for δ_n . For any $F \in C(W, \mathbf{R}^n)$, $M_n \in C(\mathbf{R}^n, X)$ and $H \in C(BX, \mathbf{R}^s)$, $M_s \in C(\mathbf{R}^s, Y)$ put $G(x) := x - M_n F(x)$ and

$$\lambda := \sup_{x \in W} \|G(x)\|_X.$$

Since W is compact in the norm $\|\cdot\|_X$ and $\delta_n(W, X) > 0$, we have $0 < \lambda < \infty$. We define $F^* \in C(W, \mathbf{R}^{n+s})$ and $M_{n+s}^* \in C(\mathbf{R}^{n+s}, Y)$ by

$$F^*(x) := (F(x), H(G(x)/\lambda)), \quad x \in W,$$

and

$$M_{n+s}^*(\xi, \eta) := M_n(\xi) + \lambda M_s(\eta), \quad \xi \in \mathbf{R}^n, \eta \in \mathbf{R}^s.$$

From the equality

$$x - M_{n+s}^* F^*(x) = \lambda \{G(x)/\lambda - M_s H(G(x)/\lambda)\}$$

and the inclusion $G(x)/\lambda \in BX$, we obtain

$$\sup_{x \in W} \|x - M_{n+s}^* F^*(x)\|_Y \leq \sup_{x \in W} \|x - M_n F(x)\|_X \sup_{x \in BX} \|x - M_s H(x)\|_Y.$$

This proves the lemma for δ_n .

For $1 \leq p \leq \infty$, \mathbf{R}^m can be equipped by the norm

$$\|x\|_{l_p^m} = \left(\sum_{k=1}^m |x_k|^p\right)^{1/p} \quad (p < \infty)$$

with the usual change to the max norm when $p = \infty$. Let B_p^m denote the unit ball in l_p^m .

Lemma 2.5. *If $1 \leq p, q \leq \infty$, $m > n$, then*

$$a_n(B_p^m, l_q^m) = \begin{cases} (n+1)^{1/q-1/p}, & \text{for } p < q, \\ 1, & \text{for } p = q, \\ (m-n)^{1/q-1/p}, & \text{for } p > q. \end{cases}$$

Proof. This lemma was proved by Tikhomirov [13] for $p = q$, by Stesin [12] for $p < q$ and by Khodulev for $p > q$. In the last case the inequality $a_n(B_p^m, l_q^m) \leq (m-n)^{1/q-1/p}$ is easily verified and the inequality $a_n(B_p^m, l_q^m) \geq (m-n)^{1/q-1/p}$ immediately follows from the first case and Lemma 2.4 by setting $X := l_p^m$, $Y := l_q^m$, $W := B_p^m$ and $s := m - n - 1$.

3. NONLINEAR WIDTHS IN BESOV SPACES

If $\alpha > 0$ and $1 \leq p, \theta \leq \infty$, then the Besov space $B_{p,\theta}^\alpha = B_{p,\theta}^\alpha(\mathbf{T}^d)$ consists of all functions f defined on \mathbf{T}^d such that the norm

$$\|f\|_{B_{p,\theta}^\alpha} := \|f\|_p + |f|_{B_{p,\theta}^\alpha}$$

is finite, where

$$|f|_{B_{p,\theta}^\alpha} := \begin{cases} (\int_0^\infty (t^{-\alpha} \omega_r(f, t)_p)^\theta dt/t)^{1/\theta}, & 1 < \theta < \infty, \\ \sup_{t \geq 0} t^{-\alpha} \omega_r(f, t)_p, & \theta = \infty, \end{cases}$$

$r > \alpha$, $\omega_r(f, \cdot)_p$ is the modulus of smoothness of order r of the function $f \in L_p(\mathbf{T}^d)$. It is well known that different values of $r > \alpha$ give equivalent norms (see, e.g., [11]). Let

$$V_m(t) = 1 + 2 \sum_{k=1}^m \cos kt + 2 \sum_{k=m+1}^{2m} ((2m - k)/m) \cos kt$$

be the de la Vallée Poussin kernel. Then the multidimensional de la Vallée Poussin sums are defined by

$$V_m(x) := \prod_{j=1}^d V_m(x_j)$$

for $m \in \mathbf{N}$. For functions f on \mathbf{T}^d consider the convolution operator $V_m f := f * V_m$ defining the de la Vallée Poussin sum of f . The differences of successive de la Vallée Poussin sums are defined by

$$\Phi_0 f := V_1 f, \quad \Phi_k := V_{2^k} f - V_{2^{k-1}} f, \quad k = 1, 2, \dots$$

If $\alpha > 0$, $1 \leq p, \theta \leq \infty$, then (cf. [11])

$$(3.1) \quad \|f\|_{B_{p,\theta}^\alpha} \approx \begin{cases} (\sum_{k=0}^\infty (2^{\alpha k} \|\Phi_k f\|_p)^\theta)^{1/\theta}, & 1 < \theta < \infty, \\ \sup_k 2^{\alpha k} \|\Phi_k f\|_p, & \theta = \infty. \end{cases}$$

Let T_m denote the set of trigonometric polynomials of order $\leq m$ in each variable x_j , $j = 1, 2, \dots, d$.

If $1 \leq p \leq \infty$, then for every $f \in T_m$

$$(3.2) \quad \|f\|_p \approx m^{-d/p} \left(\sum_{k \in P_m} |f(hk)|^p \right)^{1/p}$$

with the change to max norm when $p = \infty$, where $P_m := \{k \in \mathbf{Z}^d : -2m \leq k_j \leq 2m, j = 1, \dots, d\}$, $h = \pi/2m$ (cf. [14]). Let

$$K_{p,\theta}^\alpha := \{f \in B_{p,\theta}^\alpha : \|f\|_{B_{p,\theta}^\alpha} \leq 1\}$$

be the unit ball in $B_{p,\theta}^\alpha$. In what follows we again use the symbol γ_n for either a_n or δ_n .

Lemma 3.1. *If $\alpha > 0$, $1 \leq p, \theta, q \leq \infty$ and $\{n_k\}_{k=0}^\infty$ is a sequence of nonnegative integers such that $\sum_{k=0}^\infty n_k \leq n$, then for each nonnegative integer s*

$$\begin{aligned} \gamma_n(K_{p,\theta}^\alpha, L_q) &\ll 2^{(d/p-d/q)s} \gamma_{n_k}(B_p^{d(s+2)}, l_q^{d(s+2)}) \\ &+ \sum_{k=s+1}^\infty 2^{-(\alpha-d/p+d/q)k} \gamma_{n_k}(B_p^{d(s+2)}, l_q^{d(s+2)}). \end{aligned}$$

Proof. The lemma can be proved for both linear and Aleksandrov n -widths in similar ways. We will prove it, for example, for Aleksandrov n -widths. Moreover, because of the inclusion $B_{p,\theta'} \subset B_{p,\theta}$, $\theta' < \theta$, it is enough to prove the lemma for $\theta = \infty$.

If $f \in K_{p,\theta}^\alpha$, then

$$f = V_{2^s} f + \sum_{k=s+1}^\infty \Phi_k f,$$

the series converging in L_p , and

$$(3.3) \quad \|V_{2^s} f\|_p \leq c \quad (c > 0),$$

$$(3.4) \quad \|\Phi_k f\|_p \leq 2^{-\alpha}.$$

The inequality (3.3) follows from the inequalities $\|V_m f\|_p \leq \|V_m\|_1 \|f\|_p$ and $\|V_m\|_1 \leq c'$ ($c' > 0$) for any m , the inequality (3.4) follows from (3.1). Note that the operators V_{2^s} and $\Phi_k, k > s$, are continuous in L_p and

$$V_{2^s} f \in B_s := \{g \in T_{2^{s+1}} : \|g\|_p \leq c\}$$

and

$$\Phi_k f \in B_k := \{g \in T_{2^{k+1}} : \|g\|_p \leq 2^{-\alpha k}\}, \quad k = s + 1, s + 2, \dots$$

Let $F_k : B_k \rightarrow C_k$ be a continuous mapping from B_k into $C_k \subset T_{2^{k+1}}, k = s, s + 1, \dots$, where C_k are polyhedrons of dimension n_k . We define a polyhedron $C := \sum_{k=s}^{\infty} C_k$ of dimension $\leq n$ and the continuous mapping $F : K_{p,\infty}^\alpha \rightarrow C$ by

$$F(f) := F_s(V_{2^s} f) + \sum_{k=s}^{\infty} F_k(\Phi_k f).$$

We have for $f \in K_{p,\infty}^\alpha$

$$\|f - F(f)\|_q \leq \|V_{2^s} f - F_s(V_{2^s} f)\|_q + \sum_{k=s+1}^{\infty} \|\Phi_k f - F_k(\Phi_k f)\|_q.$$

Hence we get

$$(3.5) \quad a_n(K_{p,\infty}^\alpha, L_q) \leq a_{n_s}(B_s, T_{2^{s+1}} \cap L_q) + \sum_{k=s+1}^{\infty} a_{n_k}(B_k, T_{2^{k+1}} \cap L_q).$$

By using (3.2) we can see

$$\begin{aligned} a_{n_s}(B_s, T_{2^{s+1}} \cap L_q) &\approx 2^{(d/p-d/q)s} a_{n_s}(B_p^{d(k+s)}, l_q^{d(s+2)}), \\ a_{n_k}(B_k, T_{2^{k+1}} \cap L_q) &\approx 2^{(\alpha-d/p+d/q)k} a_{n_k}(B_p^{d(k+2)}, l_q^{d(k+2)}) \end{aligned}$$

for $k > s$. This and (3.5) prove the lemma.

Theorem 3.1. *If $1 \leq p, \theta, q \leq \infty$ and $\alpha > (d/p - d/q)_+$, then*

$$a_n(K_{p,\theta}^\alpha, L_q) \ll n^{-\alpha/d}.$$

Proof. First we consider the case $p < q$. For a given natural number n (large enough) let s be the natural number defined by the condition $2^{d(s+3)} \leq n < 2^{d(s+4)}$. Taking a fixed number δ with $0 < \delta < (\alpha - d/p + d/q)/(1/p - 1/q)$, we define the sequence $\{n_k\}_{k=s}^{\infty}$ by $n_s = 2^{d(s+2)}$ and $n_k = \lceil cn^{-2\delta d(k-s)} \rceil$ for $k > s$ with constant c depending on δ only and chosen such that $\sum_s^\infty n_k \leq n$, where $[t]$ denotes the integer part of $t \in \mathbf{R}$. To prove the theorem we now apply Lemma 3.1. Obviously,

$$a_{n_s}(B_p^{d(s+2)}, l_q^{d(s+2)}) = 0$$

and by Lemma 2.5

$$a_{n_k}(B_p^{d(k+2)}, l_q^{d(k+2)}) \approx (n2^{-\delta(k-s)})^{d/q-d/p}.$$

Hence by Lemma 3.1 we obtain

$$\begin{aligned} a_n(K_{p,\theta}^\alpha, L_q) &\ll \sum_{k=s+1}^\infty 2^{-(\alpha-d/p+d/q)k} (n2^{-\delta(k-s)})^{(d/q-d/p)} \\ &= n^{d/q-d/p} 2^{\delta s(d/q-d/p)} \sum_{k=s+1}^\infty 2^{-\beta k}, \end{aligned}$$

where $\beta = \alpha - d/p + d/q - \delta(d/p - d/q) > 0$. Therefore,

$$a_n(K_{p,\theta}^\alpha, L_q) \ll n^{d/q-d/p} 2^{\delta s(d/q-d/p)} 2^{-\beta s} \approx n^{-\alpha/d}.$$

Thus, the theorem is proved in the case $p < q$. In the case $p \geq q$ it can be proved in a similar way by setting $n_s = n$ and $n_k = 0$ for $k > s$.

Lemma 3.2. *If $1 \leq p \leq q \leq \infty$, $1 \leq \theta \leq \infty$ and $\alpha > 0$, then for any $f \in T_m$*

$$\|f\|_{B_{p,\theta}^\alpha} \ll m^\alpha \|f\|_q.$$

Proof. Since $\|f\|_p \leq (2\pi)^{d/p-d/q} \|f\|_q$ for $f \in L_q(\mathbf{T}^d)$ and $1 \leq p < q \leq \infty$, it is enough to prove the lemma for the case $p = q$. For a given natural number m (large enough), let s be the natural number such that $2^{s-1} \leq m < 2^s$. If $f \in T_m$; then $f \in T_{2^s}$ and

$$f = \sum_{k=0}^s \Phi_k f.$$

Hence, (3.1) and the inequality $\|\Phi_k f\|_p \ll \|f\|_p, k = 0, 1, 2, \dots$, give

$$\|f\|_{B_{p,\theta}^\alpha} \leq \sum_{k=0}^s 2^{\alpha k} \|\Phi_k f\|_p \ll \|f\|_p \sum_{k=0}^s 2^{\alpha k} \ll m^\alpha \|f\|_p.$$

Theorem 3.2. *If $1 \leq p \leq q \leq \infty$ or $1 < q < p < \infty$, $1 \leq \theta \leq \infty$ and $\alpha > 0$, then*

$$a_n(K_{p,\theta}^\alpha, L_q) \gg n^{-\alpha/d}.$$

Proof. If $1 \leq p \leq q \leq \infty$, then from Lemma 3.2 it follows that $b_n(K_{p,\theta}^\alpha, L_q) \gg n^{-\alpha/d}$. This and inequality (2.4) prove the theorem in the case $1 \leq p \leq q \leq \infty$. Let m be the natural number defined by the condition $(2m-1)^d < 2n \leq (2m+1)^d$. By Lemma 3.2 we have $K_{p,\theta}^\alpha \supset B := \{f \in T_m : \|f\|_p \leq cm^{-\alpha}\}$ with the constant c and, therefore

$$(3.6) \quad a_n(K_{p,\theta}^\alpha, L_q) \geq a_n(B, L_q).$$

For a function f defined on \mathbf{T}^d , let

$$(S_m f)(x) := \sum_{k \in \Delta_m} f_k e^{i\langle k, x \rangle},$$

where f_k is the k -th Fourier coefficient of f , $\langle k, x \rangle$ is the scalar product of k and x and $\Delta_m = \{k \in \mathbf{Z}^d : |k_j| \leq m, j = 1, 2, \dots, d\}$. Then S_m is a linear continuous projection from L_q onto T_m and $\|S_m\|_{L_q \rightarrow L_q} \leq c'$ ($1 < q < \infty$) for some constant c' independent of m . Applying Lemma 2.3, we get

$$(3.7) \quad a_n(B, L_q) \gg a_n(B, L_q \cap T_m).$$

By use of the Marcinkiewicz theorem (see [14] for univariate polynomials; the theorem and its proof for multivariate polynomials are completely analogous) one can verify

$$a_n(B, L_q \cap T_m) \gg m^{-\alpha} m^{(1/p-1/q)d} a_n(B_p^N, l_q^N),$$

where $N = (2m + 1)^d$. Combining this and (3.6)-(3.7), by Lemma 2.5 we prove the theorem in the case $1 < q < p < \infty$.

From Theorems 3.1-2 and 2.1 we obtain (1.3).

Remark . After submitting this paper to Proceedings of the AMS we received a manuscript by DeVore, Kyriazis, Leviatan and Tikhomirov [7] which gives some interesting results close to our paper. In particular, the authors obtained asymptotic degree of nonlinear n -widths of $K_{p,\theta}^\alpha$ in spaces L_q with $0 < q \leq \infty$, using wavelet decompositions.

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