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DERIVATIONS WITH ENGEL CONDITIONS ON MULTILINEAR POLYNOMIALS

PJEK–HWEE LEE AND TSIU–KWEN LEE

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ABSTRACT. Let R be a prime algebra over a commutative ring K with unity and let $f(X_1, \dots, X_n)$ be a multilinear polynomial over K. Suppose that d is a nonzero derivation on R such that for all $r_1, \dots, r_n$ in some nonzero ideal I of R, $\Big[d\big(f(r_1, \dots, r_n)\big), f(r_1, \dots, r_n)\Big]_k = 0$ with k fixed. Then $f(X_1, \dots, X_n)$ is central–valued on R except when char $R = 2$ and R satisfies the standard identity s_4 in 4 variables.

Throughout this note K will denote a commutative ring with unity and R will denote a prime K–algebra with center $\mathcal{Z}$. By d we always mean a nonzero derivation on R. For $x, y \in R$, set $[x, y]_0 = x$, $[x, y]_1 = [x, y] = xy - yx$ and, for $k > 1$, $[x, y]_k = \big[[x, y]_{k-1}, y\big]$.

A well–known result proved by Posner [10] states that R must be commutative if $\big[d(x), x\big] \in \mathcal{Z}$ for all $x \in R$. In [7], the authors generalized Posner's theorem by showing that a Lie ideal L of R must be contained in $\mathcal{Z}$ if char $R \neq 2$ and $\big[d(x), x\big] \in \mathcal{Z}$ for all $x \in L$. As to the case when char $R = 2$, Lanski [5] obtained the same conclusion except when R satisfies the standard identity s_4 in 4 variables. On the other hand, Vukman [11] showed that R is commutative if char $R \neq 2$ and $\big[d(x), x\big]_2 = 0$ for all $x \in R$, or if char $R \neq 2, 3$ and $\big[d(x), x\big]_2 \in \mathcal{Z}$ for all $x \in R$. In a recent paper [6], a full generalization of these results was proved by Lanski. He showed that a Lie ideal L of R is in $\mathcal{Z}$ if for some fixed $k > 0$, $\big[d(x), x\big]_k = 0$ for all $x \in L$, unless char $R = 2$ and R satisfies s_4. Note that a noncentral Lie ideal of R contains all the commutators $[x, y]$ for x, y in some nonzero ideal of R except when char $R = 2$ and R satisfies s_4. It is natural to consider the situation when $\big[d(x), x\big]_k = 0$ for all commutators $x = [x_1, x_2]$, or more generally, when $\big[d(x), x\big]_k = 0$ for all $x = f(x_1, \dots, x_n)$ where $f(X_1, \dots, X_n)$ is a multilinear polynomial over K. In the present paper, we shall extend Lanski's theorem by imposing the condition $\Big[d\big(f(x_1, \dots, x_n)\big), f(x_1, \dots, x_n)\Big]_k = 0$ on some nonzero ideal of R.

First we dispose of the simplest case when R is the matrix ring $M_m(F)$ over a field F and d is an inner derivation on R.

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Proposition. *Let F be a field and $R = M_m(F)$, the $m \times m$ matrix algebra over F. Suppose that $a \in R$ and that $f(X_1, \dots, X_n)$ is a multilinear polynomial over F such that $\big[a, f(x_1, \dots, x_n)\big]_k = 0$ for all $x_i \in R$, where $k > 0$ is a fixed integer. Then either $a \in \mathcal{Z}$ or $f(X_1, \dots, X_n)$ is central–valued on R unless* char $F = 2$ *and $m = 2$.*

Proof. If $m = 1$, there is nothing to prove, so we assume that $m \geq 2$. We assume further that either $m \neq 2$ or char $F \neq 2$ and proceed to show that $a \in \mathcal{Z}$ if $f(X_1, \dots, X_n)$ is not central–valued on R. By [8, Lemmas 2 and 9], there exists a sequence of matrices $r = (r_1, \dots, r_n)$ in R such that $f(r) = f(r_1, \dots, r_n) = \sum_{i=1}^{m} \alpha_i e_{ii}$ is a noncentral diagonal matrix, say, $\alpha_1 \neq \alpha_2$. Let φ be the inner automorphism on R defined by $x^\varphi = (1 + e_{21})x(1 - e_{21})$. Write $D = f(r)$; then $E = f(r^\varphi) = f(r_1^\varphi, \dots, r_n^\varphi) = D^\varphi = D + (\alpha_1 - \alpha_2)e_{21}$. By assumption, we have $[a, E]_k = 0$, that is, $\sum_{i=0}^{k} (-1)^i \binom{k}{i} E^i a E^{k-i} = 0$. Multiplying e_{22} from both sides, we have $(\alpha_2 - \alpha_1)^k e_{21} a e_{22} = 0$ by using $E^i = D^i + (\alpha_1^i - \alpha_2^i)e_{21}$, $e_{22}E^i = \alpha_2^i(e_{22} - e_{21}) + \alpha_1^i e_{21}$ and $E^i e_{22} = \alpha_2^i e_{22}$. Thus, $e_{21} a e_{22} = 0$ and so the $(1,2)$–entry a_{12} of a is 0. For $s \neq t$, let σ be a permutation in the symmetric group S_m such that $\sigma(1) = s$ and $\sigma(2) = t$. Let ψ be the automorphism on R defined by $x^\psi = \Big(\sum_{i,j} \xi_{ij} e_{ij}\Big)^\psi = \sum_{i,j} \xi_{ij} e_{\sigma(i),\sigma(j)}$. Then $f(r^\psi) = D^\psi$ is a diagonal matrix with (s,s)– and (t,t)–entries distinct. Replacing D by D^ψ and proceeding as before, we have $a_{st} = 0$ for $s \neq t$. In other words, a is a diagonal matrix. For any F–automorphism θ of R, a^θ enjoys the same property as a does, namely, $\big[a^\theta, f(x_1, \dots, x_n)\big]_k = 0$ for all $x_i \in R$. Hence, a^θ must be diagonal. Write $a = \sum_{i=1}^{m} a_{ii} e_{ii}$; then for $s \neq t$, we have $(1 + e_{ts})a(1 - e_{ts}) = \sum_{i=1}^{m} a_{ii} e_{ii} + (a_{ss} - a_{tt})e_{ts}$ diagonal. Hence, $a_{ss} = a_{tt}$ and so a is a scalar matrix.

We are now ready to prove the main

Theorem. *Let R be a prime K–algebra and $f(X_1, \dots, X_n)$ a multilinear polynomial over K. Suppose that d is a nonzero derivation on R such that*

$$\Big[d\big(f(x_1, \dots, x_n)\big), f(x_1, \dots, x_n)\Big]_k = 0$$

for all x_i in some nonzero ideal I of R, where $k > 0$ is fixed. Then $f(X_1, \dots, X_n)$ is central–valued on R except when char $R = 2$ *and R satisfies s_4.*

Proof. Assume first that d is Q–inner, that is, $d(x) = [a, x]$ for all $x \in R$, where a is a noncentral element in the *symmetric quotient ring* Q of R [4]. Then

$$\big[a, f(x_1, \dots, x_n)\big]_{k+1} = 0 \quad \text{for all } x_i \in I.$$

By a theorem due to Chuang [1, Theorem 2], this generalized polynomial identity $\big[a, f(X_1, \dots, X_n)\big]_{k+1}$ is also satisfied by Q. In case the center C of Q is infinite, we

have $\big[a, f(x_1, \dots, x_n)\big]_{k+1} = 0$ for all $x_i \in Q \underset{C}{\otimes} \overline{C}$ where $\overline{C}$ is the algebraic closure of C. Since both Q and $Q \underset{C}{\otimes} \overline{C}$ are prime and *centrally closed* [2, Theorems 2.5 and 3.5], we may replace R by Q or $Q \underset{C}{\otimes} \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed and $\big[a, f(x_1, \dots, x_n)\big]_{k+1} = 0$ for all $x_i \in R$.

By Martindale's theorem [9], R is then a primitive ring having a nonzero socle H with C as the associated division ring. In light of Jacobson's theorem [3, p. 75], R is isomorphic to a dense ring of linear transformations on some vector space V over C, and H consists of the linear transformations in R of finite rank. Assume first that V is finite–dimensional over C. Then the density of R on ${}_CV$ implies that $R \cong M_m(C)$ where $m = \dim_C V$. By the preceding proposition, $f(X_1, \dots, X_n)$ is central–valued on R unless char $R = 2$ and $m = 2$. Assume next that V is infinite–dimensional over C. For any $e = e^2 \in H$, we have $eRe \cong M_m(C)$ with $m = \dim_C Ve$. Since R satisfies $e\big[a, f(ex_1e, \dots, ex_ne)\big]_{k+1}e = 0$, eRe satisfies $\big[eae, f(x_1, \dots, x_n)\big]_{k+1} = 0$. As we have seen above, eae must be central in eRe if $m \geq 3$ and if $f(X_1, \dots, X_n)$ is not central–valued on eRe. Since $a \notin C$, a does not centralize the nonzero ideal H of R, so $ah_0 \neq h_0a$ for some $h_0 \in H$. Also, $f(X_1, \dots, X_n)$ is not central–valued on H, for otherwise R would satisfy the polynomial identity $\big[f(X_1, \dots, X_n), X_{n+1}\big]$ contrary to the infinite–dimensionality of ${}_CV$. So, $\big[f(h_1, \dots, h_n), h_{n+1}\big] \neq 0$ for some $h_1, \dots, h_n, h_{n+1} \in H$. By Litoff's theorem [5, p. 280], there is an idempotent $e \in H$ so that $h_0, h_0a, ah_0, h_1, \dots, h_n, h_{n+1}$ are all in eRe. Since $\dim_C V$ is infinite, we may choose e so that $m = \dim_C Ve \geq 3$. Then eae is central in eRe because $f(X_1, \dots, X_n)$ is not central–valued on eRe. Thus

$$ah_0 = eah_0 = eaeh_0 = h_0eae = h_0ae = h_0a,$$

a contradiction. Hence $f(X_1, \dots, X_n)$ must be central–valued on R except when char $R = 2$ and R satisfies s_4.

Now assume that d is not Q–inner. Recall that d can be extended uniquely to a derivation $\overline{d}$ on Q [4]. We denote by $f^d(X_1, \dots, X_n)$ the polynomial obtained from $f(X_1, \dots, X_n)$ by replacing each coefficient α with $\overline{d}\,(\alpha \cdot 1)$. Since

$$\begin{aligned}
&\Big[d\big(f(x_1, \dots, x_n)\big), f(x_1, \dots, x_n)\Big]_k \\
&\quad = \Big[f^d(x_1, \dots, x_n) + f\big(d(x_1), x_2, \dots, x_n\big) \\
&\qquad + \cdots + f\big(x_1, \dots, x_{n-1}, d(x_n)\big), f(x_1, \dots, x_n)\Big]_k = 0,
\end{aligned}$$

we have

$$\begin{aligned}
&\big[f^d(x_1, \dots, x_n) + f(y_1, x_2, \dots, x_n) \\
&\qquad + \cdots + f(x_1, \dots, x_{n-1}, y_n), f(x_1, \dots, x_n)\big]_k = 0
\end{aligned}$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in R$ by Kharchenko's theorem [4]. In particular, $\big[f^d(x_1, \dots, x_n), f(x_1, \dots, x_n)\big]_k = 0$ and $\big[f(y_1, x_2, \dots, x_n), f(x_1, \dots, x_n)\big]_k = 0$ for all $x_1, \dots, x_n, y_1 \in R$. Applying d to $\big[f(y_1, x_2, \dots, x_n), f(x_1, \dots, x_n)\big]_{2k} = 0$,

we have

$$\Big[f^d(y_1,x_2,\dots,x_n)+f\big(d(y_1),x_2,\dots,x_n\big) + \sum_{i=2}^{n} f\big(y_1,x_2,\dots,d(x_i),\dots,x_n\big), f(x_1,\dots,x_n)\Big]_{2k} + \sum_{j=0}^{2k-1}\Big[\Big[\big[f(y_1,x_2,\dots,x_n),f(x_1,\dots,x_n)\big]_j, d\big(f(x_1,\dots,x_n)\big)\Big], f(x_1,\dots,x_n)\Big]_{2k-j-1}=0.$$

The second sum vanishes since, for each j,

$$\Big[\Big[\big[f(y_1,x_2,\dots,x_n),f(x_1,\dots,x_n)\big]_j, d\big(f(x_1,\dots,x_n)\big)\Big], f(x_1,\dots,x_n)\Big]_{2k-j-1} = \sum_{i=0}^{2k-j-1}\binom{2k-j-1}{i}\Big[\big[f(y_1,x_2,\dots,x_n),f(x_1,\dots,x_n)\big]_{j+i}, \Big[d\big(f(x_1,\dots,x_n)\big), f(x_1,\dots,x_n)\Big]_{2k-j-i-1}\Big]=0$$

where either $j+i\ge k$ or $2k-j-i-1\ge k$. Thus we have

$$\Big[f^d(y_1,x_2,\dots,x_n)+f\big(d(y_1),x_2,\dots,x_n\big) + \sum_{i=2}^{n} f\big(y_1,x_2,\dots,d(x_i),\dots,x_n\big), f(x_1,\dots,x_n)\Big]_{2k}=0.$$

Again by Kharchenko's theorem, we have

$$\big[f^d(y_1,x_2,\dots,x_n)+f(z,x_2,\dots,x_n) + f(y_1,y_2,x_3,\dots,x_n)+\cdots+f(y_1,x_2,\dots,y_n), f(x_1,\dots,x_n)\big]_{2k}=0$$

for all $x_1,\dots,x_n,y_1,\dots,y_n,z\in R$. In particular,

$$\big[f^d(y_1,x_2,\dots,x_n),f(x_1,\dots,x_n)\big]_{2k}=0$$

and

$$\big[f(y_1,y_2,x_3,\dots,x_n),f(x_1,\dots,x_n)\big]_{2k}=0$$

for all $x_1,\dots,x_n,y_1,y_2\in R$. Continuing this process, we will finally get

$$\big[f(y_1,\dots,y_n),f(x_1,\dots,x_n)\big]_{nk}=0$$

for all $x_1,\dots,x_n,y_1,\dots,y_n\in R$. In light of the inner case, we have $f(y_1,\dots,y_n)\in\mathcal{Z}$ for all $y_1,\dots,y_n\in R$, that is, $f(X_1,\dots,X_n)$ is central–valued on R, except when char $R=2$ and R satisfies s_4. □

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Department of Mathematics, National Taiwan University, Taipei, Taiwan
E-mail address: phlee@math.ntu.edu.tw

Department of Mathematics, National Taiwan University, Taipei, Taiwan
E-mail address: tklee@math.ntu.edu.tw