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MULTIPLIERS OF WEAK TYPE ON LOCALLY COMPACT VILENKIN GROUPS

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ABSTRACT. Let G be a locally compact Vilenkin group with dual group Γ. We give a sufficient condition for $f \in L^\infty(\Gamma)$ to be a multiplier of weak type (p,p) on G. Some applications of our result are given. We also prove that our result is sharp.

1. INTRODUCTION

In their 1968 paper [5] on $L^p(\mathbb{R})$-multipliers, Littman, McCarthy and Rivière gave an example of a sequence of functions $(f_n)_{-\infty}^{\infty}$ such that

(i) $\operatorname{supp}(f_n) \subseteq [2^n, 2^{n+1}]$ for each $n \in \mathbb{Z}$;
(ii) $(f_n)_{-\infty}^{\infty} \in l^\infty(M_p(\mathbb{R}))$ for $1 < p < \infty$, $p \neq 2$, where $M_p(\mathbb{R})$ is the space of all $L^p(\mathbb{R})$-multipliers;
(iii) $f := \sum_{n=-\infty}^{\infty} f_n \notin M_p(\mathbb{R})$.

The above example shows that the regularity condition on each f_n in the Hörmander multiplier theorem cannot be dropped. It is natural to ask the following two questions:

Question 1. How little regularity on each f_n is needed if we only want f as defined in (iii) to be a multiplier of weak type (p,p) on $\mathbb{R}$?

Question 2. Is there a positive real number s such that the condition $(f_n)_{-\infty}^{\infty} \in l^s(M_p(\mathbb{R}))$ is itself sufficient for f as defined in (iii) to be a $L^p(\mathbb{R})$-multiplier?

In [1, Theorem 3] Carbery gave an answer to Question 1. Question 2 was answered by Cowling, Fendler and Fournier in [2, Theorem 2]. See also Onneweer and Quek [6, Theorem 1.1].

An analogue of the example by Littman et al. on locally compact Vilenkin groups G (see Definition 1.1 below) was given by Onneweer and the author in [6, Theorem 2.2]. We can therefore ask both Questions 1 and 2 mentioned earlier on G instead of on $\mathbb{R}$. In order to make our questions more precise, we give a brief description of locally compact Vilenkin groups and introduce some notation.

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Definition 1.1. A locally compact Abelian group G is said to be a locally compact Vilenkin group if there exists a strictly decreasing sequence of compact open subgroups $(G_n)_{-\infty}^{\infty}$ such that $\bigcup_{-\infty}^{\infty} G_n = G$, $\bigcap_{-\infty}^{\infty} G_n = \{0\}$ and $\sup\{\text{order } G_n/G_{n+1} : n \in \mathbb{Z}\} < \infty$.

Examples of such locally compact Vilenkin groups are the p-adic numbers and, more generally, the additive group of a local field, see Taibleson [7].

We shall denote the dual group of G by Γ, and for each $n \in \mathbb{Z}$, let Γ_n denote the annihilator of G_n, that is,

$$\Gamma_n = \{\gamma \in \Gamma : \gamma(x) = 1 \text{ for all } x \in G_n\}.$$

We choose Haar measures μ on G and λ on Γ so that $\mu(G_0) = \lambda(\Gamma_0) = 1$. Then $\mu(G_n) = (\lambda(\Gamma_n))^{-1}$ for all $n \in \mathbb{Z}$; we set $m_n := \lambda(\Gamma_n)$. For an arbitrary set A we denote its characteristic function by χ_A. The symbols $\hat{}$ and $\check{}$ will be used to denote the Fourier and inverse Fourier transform, respectively. It is easy to see that for each $n \in \mathbb{Z}$ we have

$$(\chi_{G_n})\hat{} = (\lambda(\Gamma_n))^{-1}\chi_{\Gamma_n} := F_n.$$

In order to give the definition of a (Fourier) multiplier we first introduce the space $\mathcal{S}(G)$ of test functions on G. A function $\phi : G \to \mathbb{C}$ belongs to $\mathcal{S}(G)$ if ϕ has compact support and if ϕ is constant on the cosets of some subgroup G_n (n depending on ϕ) of G. The Fourier transform maps $\mathcal{S}(G)$ one-to-one and onto $\mathcal{S}(\Gamma)$, with $\mathcal{S}(\Gamma)$ defined like $\mathcal{S}(G)$; cf. Taibleson [7, p. 37].

Let $1 \le p < \infty$. For $f \in L^\infty(\Gamma)$ and $\phi \in \mathcal{S}(G)$, define $T\phi$ by $(T\phi)\hat{} = f\hat{\phi}$. The function f is said to be

(i) an $L^p(G)$-multiplier if there exists a positive constant C so that for all $\phi \in \mathcal{S}(G)$ we have $\|T\phi\|_p \le C\|\phi\|_p$; we write $\|f\|_{M_p} = \sup\{\|T\phi\|_p/\|\phi\|_p : \phi \in \mathcal{S}(G)\}$ and denote the space of all $L^p(G)$-multipliers by $M_p(G)$;

(ii) a multiplier of weak type (p,p) on G if there exists a positive constant C so that for all $\phi \in \mathcal{S}(G)$ we have

$$\mu\{x \in G : |T\phi(x)| > t\} \le Ct^{-p}\|\phi\|_p^p, \qquad t > 0.$$

Let X be a Banach space. For $1 \le p \le \infty$ we define $l^p(X)$ to be the set of all sequences $(f_n)_{-\infty}^{\infty}$ in X such that $\sum_{n=-\infty}^{\infty} \|f_n\|_X^p < \infty$, with the usual modification if $p = \infty$.

We now state our analogues of Questions 1 and 2 on G. Let $(f_n)_{-\infty}^{\infty}$ be a sequence of functions in $L^\infty(\Gamma)$ such that $\operatorname{supp}(f_n) \subseteq \Gamma_{n+1} \setminus \Gamma_n$ for all $n \in \mathbb{Z}$.

Question 1′. How little regularity on each f_n is needed so that $\sum_{n=-\infty}^{\infty} f_n$ is a multiplier of weak type (p,p) on G?

Question 2′. Is there a positive real number s such that the condition $(f_n)_{-\infty}^{\infty} \in l^s(M_p(G))$ is itself sufficient for $\sum_{n=-\infty}^{\infty} f_n$ to be an $L^p(G)$-multiplier?

We remark that the answer to Question 2′ was obtained by Onneweer and the author in [6, Theorem 2.1]. One of our aims in this paper is to give an answer to Question 1′.

Finally, we state the results obtained in this paper. In the next section we prove the following theorem which is the main result of this paper.

Theorem 1. *Let $1 < p < 2$ and let $f \in L^\infty(\Gamma)$. Suppose*

$$\sum_{k=0}^{\infty} \sup_{i\in\mathbb{Z}} \|[(f^{i+k})\,\check{}\,\chi_{G\setminus G_i}]\,\hat{}\,\|_{M_p} < \infty,$$

where $f^n = f\chi_{\Gamma_{n+1}\setminus\Gamma_n}$. Then f is a multiplier of weak type (p,p) on G.

Our proof of Theorem 1 is motivated by Carbery [1, Theorem 1]. As a consequence of Theorem 1 we have the following corollary.

Corollary 1. *Let $\{\alpha(j)\}_1^\infty$ be a sequence of positive numbers satisfying $\sum_{j=1}^\infty j\alpha(j) < \infty$. Let $1 < p < 2$ and let $f \in L^\infty(\Gamma)$. Suppose*

$$\|f^i * (F_{j+1} - F_j)\|_{M_p} \le \alpha(i-j), \quad \text{for all } i > j,$$

where $F_j = (\chi_{G_j})\,\hat{}$ and $f^i = f\chi_{\Gamma_{i+1}\setminus\Gamma_i}$. Then f is a multiplier of weak type (p,p) on G.

We remark that Corollary 1 is an analogue on G of Theorem 3 in Carbery [1]. Thus Corollary 1 gives an answer to Question 1′.

In Section 3 we prove the following theorem which gives a simpler sufficient condition for $f \in L^\infty(\Gamma)$ to be a multiplier of weak type (p,p) on G in terms of its smoothness. We say that a function $f : \Gamma \to \mathbb{C}$ is in the Lipschitz space Λ_β, $\beta > 0$, if $f \in L^\infty(\Gamma)$ and

$$\|f\|_{\Lambda_\beta} := \sup_{\xi\neq\eta} \frac{|f(\xi) - f(\eta)|}{|\xi-\eta|^\beta} < \infty,$$

where $|\xi - \eta| = \lambda(\Gamma_n)$ if $\xi - \eta \in \Gamma_n \setminus \Gamma_{n-1}$, $n \in \mathbb{Z}$.

Theorem 2. *Let $1 < p < 2$ and let $f \in L^\infty(\Gamma)$. Suppose there exists $\beta > (2-p)/2p$ such that*

$$\|f^j\|_{\Lambda_\beta} \le C m_j^{-\beta}, \qquad j \in \mathbb{Z},$$

where C is a constant independent of j. Then f is an $L^p(G)$-multiplier.

The referee has kindly suggested the following open question whose answer seems to be beyond the methods of this paper.

Open Question. Let $1 < p < 2$ and let $\beta = (2-p)/2p$. Let $f \in L^\infty(\Gamma)$ such that

$$\|f^j\|_{\Lambda_\beta} \le C m_j^{-\beta}, \qquad j \in \mathbb{Z},$$

where C is a constant independent of j. Is f a multiplier of weak type (p,p) on G?

In the last section we show that Theorem 1 is sharp in a certain sense. More precisely, we prove the following theorem.

Theorem 3. *Let G be a dyadic group, that is,* $\operatorname{order} G_n/G_{n+1} = 2$ *for all $n \in \mathbb{Z}$. Let $1 < r < p < 2$. Then there exists a function $f \in L^\infty(\Gamma)$ such that*

$$\sum_{k=0}^{\infty} \sup_{i\in\mathbb{Z}} \|[(f^{i+k})\,\check{}\,\chi_{G\setminus G_i}]\,\hat{}\,\|_{M_p} < \infty$$

but f is not a multiplier of weak type (r,r) on G.

2. Proofs of Theorem 1 and Corollary 1

We precede the proof of Theorem 1 by two lemmas. The first lemma is an analogue on G of the Calderòn-Zygmund decomposition lemma. See Edwards and Gaudry [3, Theorem 2.3.2] for a proof of the lemma.

Lemma 2.1. *Let $\phi \in L^p(G)$, $1 \le p < \infty$, and let $\sigma > 0$ be given. Then there exists a sequence of functions $(\phi_j)_0^\infty$ such that*

(i) $\phi = \sum_{j=0}^\infty \phi_j$,
(ii) $\phi_j \in L^p(G)$ *for each* $j \ge 0$ *and* $\sum_{j=0}^\infty \|\phi_j\|_p^p \le C\|\phi\|_p^p$,
(iii) $|\phi_0(x)| \le C\sigma$ *for almost every* $x \in G$,
(iv) *there exist disjoint sets* $I_j := x_j + G_{n(j)}$ *such that* $\operatorname{supp}(\phi_j) \subseteq I_j$ *for* $j \in \mathbb{N}$,
(v) $\int_{I_j} \phi_j(x)du(x) = 0$ *for* $j \in \mathbb{N}$,
(vi) $\|\phi_j\|_p^p \le C\sigma^p \mu(I_j)$ *for* $j \in \mathbb{N}$,
(vii) $\sum_{j=1}^\infty \mu(I_j) \le C\sigma^{-p}\|\phi\|_p^p$.

Lemma 2.2. *Let $1 \le p \le 2$ and let $(h_n)_{-\infty}^\infty \in l^p(L^p(G))$. Let $\rho_n = \chi_{\Gamma_{n+1}\setminus\Gamma_n}$ for $n \in \mathbb{Z}$. Then*

$$\left\| \sum_{n=-\infty}^{\infty} (\rho_n \hat{h}_n)\check{} \right\|_p^p \le C \sum_{n=-\infty}^{\infty} \|h_n\|_p^p,$$

where C is a positive number independent of $(h_n)_{-\infty}^\infty$.

Proof. It is easy to see that for $n \in \mathbb{Z}$, $\rho_n \in M_2(G) \cap M_1(G)$ with $\|\rho_n\|_{M_2} = 1$ and $\|\rho_n\|_{M_1} = \|(\rho_n)\check{}\|_1 \le 2$. For $p = 2$ and $(h_n)_{-\infty}^\infty \in l^2(L^2(G))$, we apply Plancherel's equality to obtain

$$\begin{aligned}\left\| \sum_{n=-\infty}^{\infty} (\rho_n \hat{h}_n)\check{} \right\|_2^2 &= \left\| \sum_{n=-\infty}^{\infty} \rho_n \hat{h}_n \right\|_2^2 = \sum_{n=-\infty}^{\infty} \|\rho_n \hat{h}_n\|_2^2 \\ &= \sum_{n=-\infty}^{\infty} \|(\rho_n \hat{h}_n)\check{}\|_2^2 \le \sum_{n=-\infty}^{\infty} \|h_n\|_2^2,\end{aligned}$$

where the last inequality is due to $\|\rho_n\|_{M_2} = 1$.

For $p = 1$ and $(h_n)_{-\infty}^\infty \in l^1(L^1(G))$, we apply Minkowski's inequality to obtain

$$\left\| \sum_{n=-\infty}^{\infty} (\rho_n \hat{h}_n)\check{} \right\|_1 \le \sum_{n=-\infty}^{\infty} \|(\rho_n \hat{h}_n)\check{}\|_1 \le 2 \sum_{n=-\infty}^{\infty} \|h_n\|_1,$$

where the last inequality follows from $\|\rho_n\|_{M_1} \le 2$.

Consequently, interpolation yields for $1 < p < 2$,

$$\left\| \sum_{n=-\infty}^{\infty} (\rho_n \hat{h}_n)\check{} \right\|_p^p \le C \sum_{n=-\infty}^{\infty} \|h_n\|_p^p, \qquad (h_n)_{-\infty}^\infty \in l^p(L^p(G)),$$

which completes the proof of Lemma 2.2.

Proof of Theorem 1. We first assume that $f \in L^\infty(\Gamma)$ has compact support. For $\phi \in \mathcal{S}(G)$ define $T\phi$ by $(T\phi)\hat{} = f\hat{\phi}$. Then $T\phi = f\check{} * \phi$.

Fix $\sigma > 0$ and apply the Calderón-Zygmund decomposition to ϕ. Using the notation of Lemma 2.1 we obtain

$$\phi = \phi_0 + \sum_{j=1}^{\infty} \phi_j.$$

Let $g = \phi_0 + \sum_{j=1}^{\infty} \phi_j * \Delta_{n(j)}$, where $\Delta_{n(j)} = (\chi_{\Gamma_{n(j)}})\check{}$, and let

$$h = \sum_{j=1}^{\infty} (\phi_j - \phi_j * \Delta_{n(j)}).$$

Then

$$\begin{aligned}\{x \in G : |T\phi(x)| > \sigma\} &\subseteq \{x \in G : |Tg(x)| > \sigma/2\} \cup \{x \in G : |Th(x)| > \sigma/2\} \\ &:= E_\sigma \cup F_\sigma.\end{aligned}$$

We shall prove that $\mu(E_\sigma)$ and $\mu(F_\sigma)$ are each less than $C\sigma^{-p}\|\phi\|_p^p$ for some positive C independent of ϕ.

(a) *Estimate of* $\mu(E_\sigma)$. For the set E_σ we have

$$\mu(E_\sigma) \le 4\sigma^{-2}\|Tg\|_2^2 \le C\sigma^{-2}\|g\|_2^2 \quad \text{because } f \in M_2(G). \tag{1}$$

We claim that $\|g\|_2^2 \le C\sigma^{2-p}\|\phi\|_p^p$. We first note that

$$\Delta_{n(j)} := (\chi_{\Gamma_{n(j)}})\check{} = [\mu(G_{n(j)})]^{-1}\chi_{G_{n(j)}}.$$

Now $\operatorname{supp}(\phi_j) \subseteq I_j := x_j + G_{n(j)}$ for $j \in \mathbb{N}$. Hence we have $\operatorname{supp}(\phi_j * \Delta_{n(j)}) \subseteq I_j$. Since $I_i \cap I_j = \phi$ for $i \ne j$, $i, j \in \mathbb{N}$, we have

$$\left\| \sum_{j=1}^{\infty} \phi_j * \Delta_{n(j)} \right\|_2^2 = \sum_{j=1}^{\infty} \|\phi_j * \Delta_{n(j)}\|_2^2.$$

It follows that

$$\begin{aligned}\|g\|_2^2 &= \left\| \phi_0 + \sum_{j=1}^{\infty} \phi_j * \Delta_{n(j)} \right\|_2^2 \\ &\le C\left(\|\phi_0\|_2^2 + \sum_{j=1}^{\infty} \|\phi_j * \Delta_{n(j)}\|_2^2 \right).\end{aligned} \tag{2}$$

It is clear that

$$\begin{aligned}\|\phi_0\|_2^2 &\le \int \|\phi_0\|_\infty^{2-p} |\phi_0(x)|^p d\mu(x) \\ &\le C\sigma^{2-p}\|\phi_0\|_p^p \quad \text{(by Lemma 2.1(iii))} \\ &\le C\sigma^{2-p}\|\phi\|_p^p \quad \text{(by Lemma 2.1(ii))}.\end{aligned} \tag{3}$$

To estimate $\|\phi_j * \Delta_{n(j)}\|_2^2$ for $j \in \mathbb{N}$, we note that

$$\|\phi_j * \Delta_{n(j)}\|_p \le \|\phi_j\|_p \|\Delta_{n(j)}\|_1 = \|\phi_j\|_p$$

and

$$\|\phi_j * \Delta_{n(j)}\|_\infty \le \|\phi_j\|_p \|\Delta_{n(j)}\|_{p'} \le (C\sigma\mu(I_j)^{\frac{1}{p}})(\mu(I_j))^{-\frac{1}{p}} = C\sigma,$$

where $\frac{1}{p}+\frac{1}{p'}=1$. Thus

$$\|\phi_j * \Delta_{n(j)}\|_2^2 \le C\sigma^{2-p}\|\phi_j\|_p^p \quad \text{for } j \in \mathbb{N}.$$

It follows that

$$\begin{aligned}\sum_{j=1}^{\infty}\|\phi_j * \Delta_{n(j)}\|_2^2 &\le C\sigma^{2-p}\sum_{j=1}^{\infty}\|\phi_j\|_2^2 \\ &\le C\sigma^{2-p}\|\phi\|_p^p \quad \text{(by Lemma 2.1(ii))}.\end{aligned} \tag{4}$$

Substituting inequalities (3) and (4) into (2) we get $\|g\|_2^2 \le C\sigma^{2-p}\|\phi\|_p^p$. We now conclude that $\mu(E_\sigma) \le C\sigma^{-p}\|\phi\|_p^p$ by substituting our inequality for $\|g\|_2^2$ into (1).

(b) *Estimate of* $\mu(F_\sigma)$. We have

$$\begin{aligned}Th(x) &= \left[f\check{} * \sum_{j=1}^{\infty}(\phi_j - \phi_j * \Delta_{n(j)})\right](x) \\ &= \sum_{j=1}^{\infty} f\check{}\,\chi_{G_{n(j)}} * (\phi_j - \phi_j * \Delta_{n(j)})(x) \\ &\quad + \sum_{j=1}^{\infty} f\check{}\,\chi_{G\setminus G_{n(j)}} * (\phi_j - \phi_j * \Delta_{n(j)})(x) \\ &:= U(x) + V(x).\end{aligned} \tag{5}$$

Hence

$$\begin{aligned}F_\sigma &= \{x \in G : |Th(x)| > \sigma/2\} \\ &\subseteq \{x \in G : |U(x)| > \sigma/4\} \cup \{x \in G : |V(x)| > \sigma/4\} \\ &:= A_\sigma \cup B_\sigma.\end{aligned}$$

To estimate $\mu(A_\sigma)$ we observe that $\operatorname{supp}(\phi_j - \phi_j * \Delta_{n(j)}) \subseteq I_j := x_j + G_{n(j)}$ for $j \in \mathbb{N}$ and $\operatorname{supp}(f\check{}\,\chi_{G_{n(j)}}) \subseteq G_{n(j)}$. Thus $f\check{}\,\chi_{G_{n(j)}} * (\phi_j - \phi_j * \Delta_{n(j)})$ is supported by I_j for $j \in \mathbb{N}$ and so the function U as defined in (5) is supported by $\bigcup_{j=1}^{\infty} I_j$. It follows that $A_\sigma \subseteq \bigcup_{j=1}^{\infty} I_j$. By Lemma 2.1(vii) we have $\sum_{j=1}^{\infty}\mu(I_j) \le C\sigma^{-p}\|\phi\|_p^p$ and, consequently,

$$\mu(A_\sigma) \le C\sigma^{-p}\|\phi\|_p^p. \tag{6}$$

Next we estimate $\mu(B_\sigma)$. Let $f^i = f\chi_{\Gamma_{i+1}\setminus\Gamma_i}$ and let $\omega_i = \sum_{\{j\ge 1: n(j)=i\}}\phi_j$. Then

$$\begin{aligned}V(x) &= \sum_{j=1}^{\infty} f\check{}\,\chi_{G\setminus G_{n(j)}} * (\phi_j - \phi_j * \Delta_{n(j)})(x) \\ &= \sum_{i=-\infty}^{\infty} (f\chi_{\Gamma_i})\check{}\,\chi_{G\setminus G_i} * (\omega_i - \omega_i * \Delta_i)(x) \\ &\quad + \sum_{i=-\infty}^{\infty}\sum_{k=0}^{\infty}(f^{i+k})\check{}\,\chi_{G\setminus G_i} * (\omega_i - \omega_i * \Delta_i)(x) \\ &:= V_1(x) + V_2(x).\end{aligned}$$

Observe that $\hat{V}_1 = 0$ because $\operatorname{supp}(\omega_i - \omega_i * \Delta_i)\hat{} \subseteq \Gamma \setminus \Gamma_i$ while

$$\operatorname{supp}((f\chi_{\Gamma_i})\check{}\,\chi_{G\setminus G_i})\hat{} \subseteq \Gamma_i.$$

Hence $V_1 = 0$ and we have $B_\sigma = \{x \in G : |V_2(x)| > \sigma/4\}$.

We shall prove that $\mu(B_\sigma) \le C\sigma^{-p}\|\phi\|_p^p$ by showing that $\|V_2\|_p \le C\|\phi\|_p$. To simplify the notation let $g^{i,k} = (f^{i+k})\,\check{}\,\chi_{G\setminus G_i}$ and let $h^{i,k} = g^{i,k} * (\omega_i - \omega_i * \Delta_i)$. Then

$$\begin{aligned}\|h^{i,k}\|_p &\le \|(g^{i,k})\,\hat{}\,\|_{M_p}\|\omega_i - \omega_i * \Delta_i\|_p \\ &\le 2\|(g^{i,k})\,\hat{}\,\|_{M_p}\|\omega_i\|_p.\end{aligned}$$

Hence for fixed k we have

$$\begin{aligned}\sum_{i=-\infty}^{\infty}\|h^{i,k}\|_p^p &\le C\sup_i\|(g^{i,k})\,\hat{}\,\|_{M_p}^p\sum_{i=-\infty}^{\infty}\|\omega_i\|_p^p \\ &\le C\sup_i\|(g^{i,k})\,\hat{}\,\|_{M_p}^p\|\phi\|_p^p,\end{aligned} \tag{7}$$

where the second inequality is obtained by observing that

$$\sum_{i=-\infty}^{\infty}\|\omega_i\|_p^p = \sum_{j=-\infty}^{\infty}\|\phi_j\|_p^p$$

and by applying Lemma 2.1(ii). Consequently for fixed k we have $(h^{i,k})_{i=-\infty}^{\infty} \in l^p(L^p(G))$.

Now observe that $\operatorname{supp}(g^{i,k})\,\hat{}\,\subseteq \Gamma_{i+k+1}\setminus\Gamma_{i+k}$ and $\operatorname{supp}(\omega_i - \omega_i * \Delta_i)\,\hat{}\,\subseteq \Gamma\setminus\Gamma_i$. It is easy to see that for fixed $k \ge 0$ we have

$$\begin{aligned}\operatorname{supp}(h^{i,k})\,\hat{}\, &= \operatorname{supp}(g^{i,k})\,\hat{}\,\cap \operatorname{supp}(\omega_i - \omega_i * \Delta_i)\,\hat{}\, \\ &\subseteq \Gamma_{i+k+1}\setminus\Gamma_{i+k}.\end{aligned}$$

Now let $\rho_{i,k} = \Gamma_{i+k+1}\setminus\Gamma_{i+k}$. Then $\rho_{i,k}(h^{i,k})\,\hat{}\, = (h^{i,k})\,\hat{}$. Hence, by Lemma 2.2, we have

$$\left\|\sum_{i=-\infty}^{\infty} h^{i,k}\right\|_p^p \le C\sum_{i=-\infty}^{\infty}\|h^{i,k}\|_p^p \le C\sup_i\|(g^{i,k})\,\hat{}\,\|_{M_p}^p\|\phi\|_p^p,$$

where the last inequality follows from (7).

Applying the hypothesis of the theorem to obtain the second inequality, we have

$$\sum_{k=0}^{\infty}\left\|\sum_{i=-\infty}^{\infty} h^{i,k}\right\|_p \le C\sum_{k=0}^{\infty}\sup_i\|(g^{i,k})\,\hat{}\,\|_{M_p}\|\phi\|_p \le C\|\phi\|_p.$$

Therefore we have $\|V_2\| \le C\|\phi\|_p$ which implies that $\mu(B_\sigma) \le C\sigma^{-p}\|\phi\|_p^p$. Combining with our estimate of $\mu(A_\sigma)$ in (6) we conclude that $\mu(F_\sigma) \le C\sigma^{-p}\|\phi\|_p^p$. It follows from our estimates of $\mu(E_\sigma)$ and $\mu(F_\sigma)$ that for $\sigma > 0$, we have

$$\mu\{x \in G : |T\phi(x)| > \sigma\} \le C\sigma^{-p}\|\phi\|_p^p \tag{8}$$

for all $\phi \in \mathcal{S}(G)$. Hence we have proved that f is a multiplier of weak type (p,p) on G under the additional assumption that f has compact support.

In general, for $f \in L^\infty(\Gamma)$ satisfying the hypothesis of Theorem 1 and for given $\phi \in \mathcal{S}(G)$, there exists $k \in \mathbb{Z}$ such that $T\phi = T_k\phi$ where $(T_k\phi)\,\hat{}\, = (f\chi_{\Gamma_k})\hat{\phi}$. Since $f\chi_{\Gamma_k}$ also satisfies the hypothesis of Theorem 1, it follows from (8) that

$$\mu\{x \in G : |T_k\phi(x)| > \sigma\} \le C\sigma^{-p}\|\phi\|_p^p.$$

An examination of our proof of (8) shows that the constant C is independent of k. Hence we have

$$\mu\{x \in G : |T\phi(x)| > \sigma\} = \mu\{x \in G : |T_k\phi(x)| > \sigma\} \leq C\sigma^{-p}\|\phi\|_p^p.$$

The proof of Theorem 1 is now complete.

Proof of Corollary 1. To simplify the notation, we write $g^{i,k} = (f^{i+k})\check{}\,\chi_{G\setminus G_i}$ and $g_l^{i,k} = (f^{i+k})\hat{}\,\chi_{G_{i+l}\setminus G_{i+l+1}}$. Then $g^{i,k} = \sum_{l=-\infty}^{-1} g_l^{i,k}$ and $(g_l^{i,k})\hat{} = f^{i+k} * (F_{i+l+1} - F_{i+l})$, where $F_n := (\chi_{G_n})\hat{}$. For $i \in \mathbb{Z}$ and $k \geq 0$, we have

$$\begin{aligned}
\|(g^{i,k})\hat{}\,\|_{M_p} &\leq \sum_{l=-\infty}^{-1} \|(g_l^{i,k})\hat{}\,\|_{M_p} \\
&= \sum_{l=-\infty}^{-1} \|f^{i+k} * (F_{i+l+1} - F_{i+l})\|_{M_p} \\
&\leq \sum_{l=-\infty}^{-1} \alpha(k-l).
\end{aligned}$$

Hence

$$\sum_{k=0}^{\infty} \sup_i \|(g^{i,k})\hat{}\,\|_{M_p} \leq \sum_{k=0}^{\infty}\sum_{l=-\infty}^{-1} \alpha(k-l) \leq \sum_{j=1}^{\infty} j\alpha(j) < \infty.$$

Consequently the hypothesis of Theorem 1 is satisfied and we conclude that f is a multiplier of weak type (p,p) on G.

3. Proof of Theorem 2

Let $k \geq 0$ and $l < 0$. Using the notation in the proof of Corollary 1, we have

$$\begin{aligned}
\|(g_l^{i,k})\|_2 &= \|(g_l^{i,k})\hat{}\,\|_2 \\
&= \|f^{i+k} * (F_{i+l+1} - F_{i+l})\|_2 \\
&= \|[f^{i+k} * (F_{i+l+1} - F_{i+l})]\chi_{\Gamma_{i+k+1}\setminus\Gamma_{i+k}}\|_2 \quad (\text{since } k > l) \\
&\leq \|f^{i+k} * (F_{i+l+1} - F_{i+l})\|_\infty \|\chi_{\Gamma_{i+k+1}\setminus\Gamma_{i+k}}\|_2 \\
&\leq C\|f^{i+k}\|_{\Lambda_\beta}(m_{i+l+1})^\beta (m_{i+k+1})^{\frac{1}{2}} \\
&\leq C(m_{i+k})^{-\beta}(m_{i+l+1})^\beta (m_{i+k+1})^{\frac{1}{2}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|(g_l^{i,k})\hat{}\,\|_{M_1} &= \|g_l^{i,k}\|_1 \\
&\leq \|g_l^{i,k}\|_2 \|\chi_{G_{i+l}\setminus G_{i+l+1}}\|_2 \\
&\leq C[(m_{i+k})^{-\beta}(m_{i+l+1})^\beta (m_{i+k+1})^{\frac{1}{2}}](m_{i+l})^{-\frac{1}{2}} \\
&\leq C(m_{i+k})^{-\beta+\frac{1}{2}}(m_{i+l})^{\beta-\frac{1}{2}}.
\end{aligned}$$

We also have

$$\|(g_l^{i,k})\hat{}\,\|_{M_2} = \|(g_l^{i,k})\hat{}\,\|_\infty \leq \|f^{i+k} * (F_{i+l+1} - F_{i+l})\|_\infty \leq C(m_{i+k})^{-\beta}(m_{i+l})^\beta.$$

Interpolation yields for $1 < q < 2$,

$$\|(g_l^{i,k})\hat{}\,\|_{M_q} \leq C(m_{i+l})^{\beta-(2-q)/2q}(m_{i+k})^{(2-q)/2q-\beta}.$$

Now choose q such that $1 < q < p$ and $\beta > (2-q)/2q$. Then we have

$$\begin{aligned}\sup_i \|(g^{i,k})\,\hat{}\,\|_{M_q} &\le \sup_i \left\{ \sum_{l=-\infty}^{-1} \|(g_l^{i,k})\,\hat{}\,\|_{M_q} \right\} \\ &\le \sup_i \{ C m_i^{\beta-(2-q)/2q} (m_{i+k})^{(2-q)/2q-\beta} \} \\ &\le C(2^{-k})^{\beta-(2-q)/2q},\end{aligned}$$

where the last inequality follows from the fact that $m_{i+k} \ge 2^k m_i$ for all $i \in \mathbb{Z}$.

It is now clear that

$$\sum_{k=0}^{\infty} \sup_i \|(g^{i,k})\,\hat{}\,\|_{M_q} \le C \sum_{k=0}^{\infty} (2^{-k})^{\beta-(2-q)/2q} < \infty.$$

By Theorem 1 we have f a multiplier of weak type (q,q) on G. Since $1 < q < p < 2$ and f is also an $L^2(G)$-multiplier, we have f an $L^p(G)$-multiplier.

4. Proof of Theorem 3

In this section G will denote a dyadic group, that is, order $G_n/G_{n+1} = 2$ for all $n \in \mathbb{Z}$. Let $1 < r < p < 2$. Choose q such that $r < q < p$ and choose α such that $0 < \alpha < 1$, $q < 2/(2-\alpha) < p$. As in Gaudry and Inglis [4, Example 5.2] we construct Rudin-Shapiro-like polynomials on G as follows:

For $n \ge 0$, fix γ_0^n in $\Gamma_{2n+2} \setminus \Gamma_{2n+1}$. Then for $k = 1, \dots, n+1$, set

$$\begin{aligned}\rho_0^n &= \sigma_0^n = \chi_{G_n}\gamma_0^n, \\ \rho_k^n &= \rho_{k-1}^n + \gamma_k^n \sigma_{k-1}^n, \\ \sigma_k^n &= \rho_{k-1}^n - \gamma_k^n \sigma_{k-1}^n,\end{aligned}$$

where γ_k^n are chosen from Γ_{2n+1} such that $(\rho_k^n)\,\hat{}$ and $(\sigma_k^n)\,\hat{}$ are both constant and non-zero on precisely 2^k cosets of Γ_n in $\Gamma_{2n+2} \setminus \Gamma_{2n+1}$. Now define Ω on Γ by

$$\Omega(\gamma) = \begin{cases} \operatorname{sgn}(\rho_{n+1}^n)\,\hat{}\,(\gamma) & \text{if } \gamma \in \Gamma_{2n+2} \setminus \Gamma_{2n+1},\ n \ge 0, \\ 0 & \text{otherwise.}\end{cases}$$

Define f on Γ by

$$f(\gamma) = \sum_{n=0}^{\infty} 2^{(\alpha-1)n/4} \chi_{\Gamma_n \setminus \Gamma_{n-1}}(\gamma)\Omega(\gamma).$$

Note that for $n \ge 0$, $f(\gamma)$ is constant $(= \pm 2^{(\alpha-1)(n+1)/2})$ on the 2^{n+1} cosets of Γ_n in $\Gamma_{2n+2} \setminus \Gamma_{2n+1}$ and is zero elsewhere. Let $f^j = f\chi_{\Gamma_{j+1}\setminus\Gamma_j}$. Then for $n \ge 1$, we have $|f^{2n+1}(\gamma) - f^{2n+1}(\rho)| \le 2 \cdot 2^{(\alpha-1)(n+1)/2}$, where $\gamma, \rho \in \Gamma$. Hence

$$\begin{aligned}\|f^{2n+1}\|_{\Lambda_{(1-\alpha)/2}} &\le \sup_{\gamma \ne \rho} \frac{|f^{2n+1}(\gamma) - f^{2n+1}(\rho)|}{|\gamma - \rho|^{(1-\alpha)/2}} \\ &\le 2^{1+(\alpha-1)(n+1)/2 - n(1-\alpha)/2} \\ &\le C \cdot 2^{-2n(1-\alpha)/2}.\end{aligned}$$

It is easy to see that $f^j = 0$ for $j \ne 2n+1$, $n \ge 1$. Hence f satisfies the hypothesis of Theorem 2 with $\beta = (1-\alpha)/2 > (2-p)/2p$ which implies that f also satisfies the hypothesis of Theorem 1 for $p > 2/(2-\alpha)$. Thus f is a multiplier of weak type (p,p) on G. But by Gaudry and Inglis [4, Example 5.2], f is not an $L^q(G)$-multiplier.

Since $r < q < p$, f is not a multiplier of weak type (r, r) on G because otherwise f is an $L^q(G)$-multiplier, by interpolation between r and p.

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