ON THE HIGHER DELTA INVARIANTS OF A GORENSTEIN LOCAL RING

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To the memory of Professor Maurice Auslander

Abstract. Let \((R, m)\) be a Gorenstein complete local ring. Auslander’s higher delta invariants are denoted by \(\delta_{R}^{n}(M)\) for each module \(M\) and for each integer \(n\). We propose a conjecture asking if \(\delta_{R}^{n}(R/m^\ell) = 0\) for any positive integers \(n\) and \(\ell\). We prove that this is true provided the associated graded ring of \(R\) has depth not less than \(\dim R - 1\). Furthermore we show that there are only finitely many possibilities for a pair of positive integers \((n, \ell)\) for which \(\delta_{R}^{n}(R/m^\ell) > 0\).

0. Introduction

Let \((R, m, k)\) be a commutative Noetherian Gorenstein complete local ring. Maurice Auslander has introduced the delta invariant \(\delta_{R}(M)\) for a finitely generated \(R\)-module \(M\). It is defined to be the smallest integer \(\mu\) such that there is an epi-morphism \(X \oplus R^\mu \longrightarrow M\) with \(X\) a maximal Cohen-Macaulay module with no free summands. For an integer \(n \geq 0\), the \(n\)th delta invariant \(\delta_{R}^{n}(M)\) is also defined by Auslander, Ding and Solberg \([2]\) as \(\delta_{R}^{n}(M) = \delta_{R}(\Omega_{R}^{n}(M))\), where \(\Omega_{R}^{n}(M)\) denotes the \(n\)th syzygy module of \(M\) over \(R\). On the other hand, S. Ding ([3], [4] and [5]) studies the delta invariants of \(R/m^\ell\) \((\ell \geq 1)\) and defines an interesting, new invariant - the index of \(R\).

In this paper we are particularly interested in the higher delta invariants of \(R/m^\ell\), that is, \(\delta_{R}^{n}(R/m^\ell)\) for \(n, \ell \geq 1\). We would like to propose the following conjecture:

Conjecture (0.1). For any positive integers \(n\) and \(\ell\), we would have \(\delta_{R}^{n}(R/m^\ell) = 0\) unless \(R\) is regular.

Actually, as one of the main theorems of this paper, we shall show this conjecture is true if the associated graded ring \(gr_m(R)\) has depth \(\geq \dim R - 1\). See Corollary (2.3). In particular, it is valid if \(R\) is either a ring of hypersurface or a homogeneous graded ring. We note that, if \(d = \dim(R)\), then that \(\delta_{R}^{d}(R/m^\ell) = 0\) exactly means that the \(d\)th syzygy module of \(R/m^\ell\) has no free direct summand. Hence our result generalizes a result of Herzog \([6, \text{Cor (2.4)}]\).
Furthermore we can show in Theorem (2.1) that, in general, there are only finitely many possibilities for a pair \((n, \ell)\) of positive integers for which \(\delta^n(R/m^\ell) > 0\).

1. Some preliminaries

In this paper \((R, m, k)\) will always be a Gorenstein complete local ring. We recall some basic facts on the delta invariants from [1] and [2].

For a finitely generated \(R\)-module \(M\), an exact sequence

\[
0 \longrightarrow Y \xrightarrow{f} X \longrightarrow M \longrightarrow 0
\]

is called a Cohen-Macaulay approximation of \(M\) if \(X\) is a maximal Cohen-Macaulay module and \(Y\) has finite projective dimension. We say that the sequence is minimal if there are no isomorphisms split out of \(f\). It is known that a minimal Cohen-Macaulay approximation of \(M\) exists uniquely up to isomorphisms.

An \(R\)-module is said to be stable if there is no nontrivial free direct summand. Since \(R\) is a complete local ring, every finitely generated \(R\)-module \(X\) is uniquely decomposed as a direct sum of a stable module and a free module. We denote the maximal rank of free direct summand of \(X\) by \(f\text{-rank}_R X\). If the sequence (1.1) is the minimal Cohen-Macaulay approximation of \(M\), then the delta invariant \(\delta_R(M)\) is defined as \(f\text{-rank}_R X\). It is known that \(\delta_R(M) = 0\) if and only if \(M\) is a homomorphic image of a stable maximal Cohen-Macaulay module.

For an integer \(n\), the \(n\)th delta invariant \(\delta^n_R(M)\) is, by definition, the delta invariant of the \(n\)th syzygy module \(\Omega^n_R(M)\) of \(M\). Note that if \(n > \dim R\), then \(\delta^n_R(M) = 0\) for any finitely generated module \(M\), since \(\Omega^n_R(M)\) is a stable maximal Cohen-Macaulay module.

Auslander has shown the following

**Lemma (1.2)** ([2, Proposition 5.7]). If \(R\) is non-regular, then \(\delta^0_R(k) = 0\) for any \(n \geq 0\).

As a result of this lemma we have

**Lemma (1.3).** Suppose that \(R\) is non-regular and let \(M\) be a finitely generated \(R\)-module. Assume that \(\phi: R \longrightarrow R'\) is a faithfully flat morphism and \(\dim R = \dim R'\). Then \(\delta^1_R(mM) = 0\).

**Proof.** Since \(\delta_R(m) = \delta^1_R(R/m) = 0\), there is an epimorphism \(X \longrightarrow m\) with \(X\) a stable maximal Cohen-Macaulay module. If \(F \longrightarrow M\) is a free cover of \(M\), then we have an epimorphism \(F \odot_R X \longrightarrow mM\), where \(F \odot_R X\) is also a stable maximal Cohen-Macaulay module. Thus \(\delta_R(mM) = 0\).

**Corollary (1.4).** Conjecture (0.1) is true if \(\dim R \leq 1\).

We now remark on delta invariants under some ring extension.

**Lemma (1.5).** Let \(\phi: R \longrightarrow R'\) be a local homomorphism of Gorenstein complete local rings and let \(M\) be a finitely generated \(R\)-module. Assume that \(\phi\) is a faithfully flat morphism and \(\dim R = \dim R'\). Let the minimal Cohen-Macaulay approximation of \(M\) over \(R\) be given as (1.1). Then the sequence

\[
0 \longrightarrow Y \odot_R R' \xrightarrow{f \odot_R} X \odot_R R' \longrightarrow M \odot_R R' \longrightarrow 0
\]

is the minimal Cohen-Macaulay approximation over \(R'\).
Proof. Since \( R' \) is \( R \)-flat and has the same dimension as \( R \), we can show that if \( X \) is a maximal Cohen-Macaulay module over \( R \) (resp. has finite projective dimension over \( R \)), then so is \( X \otimes R R' \) over \( R' \). Thus the sequence (1.6) is a Cohen-Macaulay approximation of \( M \otimes R R' \) over \( R' \). We have to show that (1.6) is minimal. Suppose not. Then we would have an \( R' \)-homomorphism \( g : X \otimes R R' \rightarrow R' \) such that the composition \( g \cdot (f \otimes R') \) is an epimorphism. Since \( \text{Hom}_{R'}(X \otimes R R', R') \cong \text{Hom}_R(X, R) \otimes_R R' \), we can write \( g \) as a finite sum of \( h_i \otimes R' \) with \( h_i \in \text{Hom}_R(X, R) \). Since \( R \) and \( R' \) are local rings, there is an \( i \) such that \( (h_i \cdot f) \otimes R' \), hence \( h_i \cdot f \), is an epimorphism. This contradicts the sequence (1.1) being minimal. \( \square \)

**Lemma (1.7).** Under the same notation as in (1.5), we have the equality

\[
\delta^n_{R}(M) = \delta^n_{R'}(M \otimes_R R'),
\]

for any integer \( n \geq 0 \).

Proof. Since \( R' \) is \( R \)-flat, we can see that \( \Omega^n_{R'}(M \otimes_R R') = \Omega^n_{R'}(M) \otimes_R R' \). Thus applying Lemma (1.5), we have only to show that \( f \)-rank\(_R(X) = f \)-rank\(_R(X \otimes R R') \) if \( X \) is a maximal Cohen-Macaulay module over \( R \). For this, it is enough to prove that if \( X \) is a stable \( R \)-module, then \( X \otimes R R' \) is also stable as an \( R' \)-module. Suppose \( X \otimes R R' \) is not stable over \( R' \). Then we have an epimorphism \( g : X \otimes R R' \rightarrow R' \). As in the proof of (1.5) we can write \( g \) as a finite sum of \( h_i \otimes R' \) with \( h_i \in \text{Hom}_R(X, R) \). Then \( h_i \) is an epimorphism for some \( i \). This contradicts that \( X \) is stable. \( \square \)

The same equality as in (1.7) is discussed in a recent work of Shida [7]. The following lemma is necessary for the proof of our theorem:

**Lemma (1.8).** Let \( x \in \mathfrak{m} \) be a non zero-divisor both on \( R \) and on a finitely generated \( R \)-module \( M \). Putting \( \overline{R} = R/xR \), we have the equality \( \delta^n_{\overline{R}}(M \otimes_R \overline{R}) = \delta_R(M) \).

This lemma follows easily from [2, Lemma 5.1].

2. Main theorems

In the rest of the paper, \((R, \mathfrak{m}, k)\) is a complete Gorenstein local ring of dimension \( d \), and we always assume that \( R \) is non-regular. Our main theorems in this paper are the following:

**Theorem (2.1).** There is an integer \( \ell_0 \) such that \( \delta^n_{R}(R/\mathfrak{m}^{\ell}) = 0 \) for any \( \ell \geq \ell_0 \) and for any \( n > 0 \).

**Theorem (2.2).** Let \( G = \text{gr}_\mathfrak{m}(R) \) be the associated graded ring of \( R \) with respect to the maximal ideal. If depth \( G = t \), then we have \( \delta^n_{R}(R/\mathfrak{m}^{\ell}) = 0 \) for any \( \ell > 0 \) and for any \( n \geq d - t + 1 \).

As a consequence of the above theorems, we see that there are only finitely many possibilities for a pair of positive integers \((n, \ell)\) with \( \delta^n_{R}(R/\mathfrak{m}^{\ell}) > 0 \). Furthermore we have the following corollary as a direct consequence of (2.2), which generalizes a result of Herzog [6, Cor. (2.4)].

**Corollary (2.3).** Suppose that the associated graded ring \( G \) has depth \( \geq d - 1 \). (For example, \( R \) is either a ring of hypersurface or a homogeneous graded ring.) Then \( \delta^n_{R}(R/\mathfrak{m}^{n}) = 0 \) for any positive integers \( n \) and \( \ell \).

Proof. Apply (2.2) to get \( \delta^n_{R}(R/\mathfrak{m}^{n}) = 0 \) for \( n \geq 2 \), then combine it with (1.3). \( \square \)

Before proceeding to the proof, we make the following
Remark 2.4. When proving Theorems (2.1) and (2.2), we may assume that the residue field \( k \) is an infinite field.

To show this, let \( u \) be an indeterminate and let \( R' \) be the completion of the local ring \( R[u]/(u) \). Then \( R' \) is a faithfully flat \( R \)-algebra, which is a complete Gorenstein local ring with maximal ideal \( \mathfrak{m}' = \mathfrak{m}R' \) and with residue field \( k' = k(u) \) that is, in fact, an infinite field. Notice in this setting that \( R'/\mathfrak{m}'k' = R/\mathfrak{m} \) and \( R'/\mathfrak{m}'k' \). Hence it follows from (1.7) that \( \delta_{R'}(R'/\mathfrak{m}') = \delta_{R}(R/\mathfrak{m}) \). Note also that \( \text{gr}_{\mathfrak{m}'}(R') = \text{gr}_{\mathfrak{m}}(R) \otimes_R R' \), in particular, one can show depth \( \text{gr}_{\mathfrak{m}'}(R') = \text{depth} \text{gr}_{\mathfrak{m}}(R) \). Thus, if necessary, taking \( R' \) instead of \( R \), we may assume that \( R \) has an infinite residue field.

We need a lemma to prove the theorems.

**Lemma (2.5).** Let \( M \) be a finitely generated \( R \)-module and let \( x \in \mathfrak{m} \) be a non-zero divisor on \( R \). Suppose that the initial form \( x^+ \) of \( x \) in \( G = \text{gr}_{\mathfrak{m}}(R) \) is a non-zero divisor on \( \text{gr}_{\mathfrak{m}}(M) \). Furthermore we denote \( \overline{R} = R/xR \) and \( \overline{M} = M/xM \). Then we have the following isomorphism for each \( n \geq 0 \):

\[
\Omega^n_R(\mathfrak{m}M) \otimes_R \overline{R} \cong \Omega^n_{\overline{R}}(\mathfrak{m}\overline{M}) \oplus \Omega^n_{\overline{R}}(M/M\mathfrak{m}M).
\]

**Proof.** We note from the assumption that \( x \) is a non-zero divisor on \( M \) and the multiplication map \( M/\mathfrak{m}M \rightarrow M/M^2M \) of \( x \) is an injective map. Thus we have the following commutative diagram with an exact row:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & M/\mathfrak{m}M & \xrightarrow{x} & \mathfrak{m}M/x\mathfrak{m}M & \longrightarrow & M/xM & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \text{natural map} & & & \\
0 & \longrightarrow & M/\mathfrak{m}M & \xrightarrow{x} & M/\mathfrak{m}M^2 & & & \\
\end{array}
\]

As the map in the second row is an injective map of \( k \)-vector spaces, it is split and thus the first row is a split exact sequence. Therefore we see that

\[
\mathfrak{m}M \otimes_R \overline{R} \cong \mathfrak{m}(M/xM) \oplus M/\mathfrak{m}M.
\]

Now we prove the lemma by induction on \( n \). For \( n = 0 \) the lemma is nothing but the above isomorphism. For \( n > 0 \), we take the minimal free cover of \( \text{gr}_{\mathfrak{m}}(\mathfrak{m}M) \) to get the exact sequence:

\[
0 \longrightarrow \Omega^n_R(\mathfrak{m}M) \longrightarrow F \longrightarrow \Omega^{n-1}_R(\mathfrak{m}M) \longrightarrow 0.
\]

Taking the tensor product of this sequence with \( \overline{R} \), we can show that the sequence

\[
0 \longrightarrow \Omega^n_R(\mathfrak{m}M) \otimes_R \overline{R} \longrightarrow F \otimes_R \overline{R} \longrightarrow \Omega^{n-1}_R(\mathfrak{m}M) \otimes_R \overline{R} \longrightarrow 0
\]

gives a minimal free cover over \( \overline{R} \). Thus we have an isomorphism

\[
\Omega^n_R(\mathfrak{m}M) \otimes_R \overline{R} \cong \Omega^n_{\overline{R}}(\Omega^{n-1}_R(\mathfrak{m}M) \otimes_R \overline{R}).
\]

The lemma follows from this isomorphism together with the induction hypothesis. \( \square \)

We should mention that a reduction argument similar to that used in this lemma already appeared in Ding [5].

**Proof of Theorem (2.1).** In this proof we assume that \( k \) is an infinite field by (2.4). We prove the theorem by induction on \( d = \text{dim} R \). If \( d \leq 1 \), then the theorem holds by (1.4).
Now suppose $d > 1$ and we denote the associated graded ring of $R$ with respect to the maximal ideal by $G = gr_m(R)$ and the irrelevant maximal ideal of $G$ by $G_I$. Let $I = \mathcal{H}^0_{G_I}(G)$ be the ideal of $G$ consisting of all elements which are annihilated by some powers of $G_I$. Note that $I$ is a graded ideal of finite length, hence there is an integer $\ell_1$ such that $I\ell = (0)$ for any $\ell \geq \ell_1$. Since $G/I$ has positive depth and since $k$ is an infinite field, we can find an element $x \in m - m^2$ so that the initial form $x^* \in G$ is a non-zero divisor on $G/I$. Then, for $\ell \geq \ell_1$, the multiplication of $x$ induces the injective map

$$m^\ell/m^\ell+1 \rightarrow m^\ell+1/m^\ell+2,$$

since $(G/I)_\ell = G_\ell$. In particular, one sees that $x^*$ is a non-zero divisor on the $G$-module $gr_m(m^\ell)$ for $\ell \geq \ell_1$. Thus we can apply Lemma (2.5) to get the isomorphism:

$$(2.6) \quad \Omega^n_{\mathcal{R}}(m^\ell+1) \otimes_R \mathcal{R} \cong \Omega^n_{\mathcal{R}}(m^\ell+1/xm^\ell) \oplus \Omega^n_{\mathcal{R}}(m^\ell/m^\ell+1) \quad (\ell \geq \ell_1),$$

where $\mathcal{R} = R/xR$.

Let $\overline{m}$ be the maximal ideal $m/xR$ of $\mathcal{R}$. We shall show that there is an integer $\ell_2(\geq \ell_1)$ such that $\overline{m}^\ell+1 = m^\ell+1/xm^\ell$ for any $\ell \geq \ell_2$. To show this, it is enough to prove $xR \cap m^\ell+1 = xm^\ell$ for $\ell \geq \ell_2$, since $\overline{m}^\ell+1 = m^\ell+1/xR \cap m^\ell+1$. By the lemma of Artin-Rees, we know that there is an integer $r (> 0)$ such that $xR \cap m^\ell+1 = m^\ell+1-r(xR \cap m^r)$ for $\ell \geq r$. Hence

$$(2.7) \quad xR \cap m^\ell+1 = xm^{\ell+1-r} \cap m^\ell+1,$$

for such $\ell$. Now take $\ell$ such that $\ell \geq \ell_1 + r - 1$. Then, since $\ell + i - r \geq \ell_1$ for $i \geq 1$, we see that $(G/I)_{\ell+i-r} = G_{\ell+i-r}$ for $i \geq 1$. Thus the multiplication by $x$ induces the following sequence of injective maps:

$$m^{\ell+1-r}/m^{\ell+2-r} \rightarrow m^{\ell+2-r}/m^{\ell+3-r} \rightarrow \cdots \rightarrow m^{\ell-1}/m^{\ell} \rightarrow m^{\ell}/m^{\ell+1}.$$ 

This means that the right hand side of (2.7) equals $xm^\ell$. Therefore we have shown the equality $\overline{m}^\ell+1 = m^\ell+1/xm^\ell$ holds for $\ell \geq \ell_2 := \ell_1 + r - 1$. Combining this with (2.6), we thus have

$$(2.8) \quad \Omega^n_{\mathcal{R}}(m^\ell+1) \otimes_R \mathcal{R} \cong \Omega^n_{\mathcal{R}}(m^\ell+1) \oplus \Omega^n_{\mathcal{R}}(m^\ell/m^\ell+1),$$

for $\ell \geq \ell_2$.

Now by the induction hypothesis, we find an integer $\ell_3$ such that

$$\delta^n_{\mathcal{R}} (\Omega^n_{\mathcal{R}}(m^\ell+1)) = \delta^n_{\mathcal{R}}(\overline{m}^\ell+1) = 0,$$

for $\ell \geq \ell_3$. On the other hand, we know from (1.2) that $\delta^n_{\mathcal{R}}(m^\ell/m^\ell+1) = 0$ for any $\ell$ and $n$. Thus it follows from (2.8) that $\delta^n_{\mathcal{R}} (\Omega^n_{\mathcal{R}}(m^\ell+1) \otimes_R \mathcal{R}) = 0$ for $\ell \geq \ell_0 := \max\{\ell_2, \ell_3\}$. Then it follows from (1.8) that $\delta^n_{\mathcal{R}}(m^\ell+1) = \delta_{\mathcal{R}} (\Omega^n_{\mathcal{R}}(m^\ell+1)) = 0$ for $\ell \geq \ell_0$ as desired. 

\textbf{Proof of Theorem (2.2).} The proof of this theorem goes through in the same way as the proof of (2.1). We may assume that $k$ is an infinite field. We prove the theorem by induction on $t = \text{depth} G$. If $t = 0$, then the theorem obviously holds since $\delta^n_{\mathcal{R}}(M) = 0$ for any $n > d = \text{dim} R$ and for any finitely generated $R$-module $M$. Assume $t \geq 1$. Since $k$ is infinite, there is an element $x \in m - m^2$ such that the initial form $x^* \in G$ is a non-zero-divisor on $G$. For a fixed integer $\ell$, we can apply
Lemma (2.5) to $M = \mathfrak{m}^{\ell-1}$, since $x^\ell$ is also a non-zero-divisor on $gr_m(\mathfrak{m}^{\ell-1}) \subseteq G$. Thus we have an isomorphism for each $n > 0$:

$$(2.9) \quad \Omega_{R}^{n-1}(m^\ell) \otimes_R \overline{R} \cong \Omega_{\overline{R}}^{n-1}(\overline{m}^\ell) \oplus \Omega_{\overline{R}}^{n-1}(m^{\ell-1}/m^\ell),$$

where $\overline{R} = R/xR$ and $\overline{m} = m/xR$. Note in this case that we have $m^{\ell+1}/xm^\ell = m^{\ell+1}$ for any $\ell \geq 0$, since $x^\ell$ is a non-zero divisor on $G$. Compare with the proof of (2.1).

We also notice that $gr_m(\overline{R}) = G/x^*G$, in particular, depth $gr_m(\overline{R}) = t-1$. Thus by the induction hypothesis, we see that

$$\delta_{\overline{R}} \left( \Omega_{\overline{R}}^{n-1}(\overline{m}^\ell) \right) = \delta_{\overline{R}} \left( \Omega_{\overline{R}}^{n}(\overline{R}/\overline{m}^\ell) \right) = \delta_{\overline{R}}(\overline{R}/\overline{m}^\ell) = 0,$$

for any $n \geq (d-1)-(t-1)+1 = d-t+1$. On the other hand, we know from (1.2) that $\delta_{\overline{R}} \left( \Omega_{\overline{R}}^{n-1}(m^{\ell-1}/m^\ell) \right) = \delta_{\overline{R}}^{n-1}(m^{\ell-1}/m^\ell) = 0$ for any $n > 0$. Therefore it follows from (2.9) that $\delta_{\overline{R}} \left( \Omega_{\overline{R}}^{n-1}(m^\ell) \otimes_R \overline{R} \right) = 0$ if $n \geq d-t+1$. Thus Lemma (1.8) implies that $\delta_{\overline{R}}^{n}(R/m^\ell) = \delta_{\overline{R}}^{n-1}(m^\ell) = \delta_{\overline{R}}(\Omega_{\overline{R}}^{n-1}(m^\ell)) = 0$ as desired. \hfill \Box

3. Some remarks for two-dimensional cases

In this section we assume that the Gorenstein complete local ring $(R, m, k)$ has dimension 2. We denote the associated graded ring $gr_m(R)$ by $G$, as before.

As we have shown in (2.2), if $G$ has positive depth, then the conjecture (0.1) is true for $R$. However there is, of course, a Gorenstein local ring $R$ with depth $G = 0$. One of the easiest examples is

$$R = k[[u, v, w, x, y]]/(u^2, uw - v^3, uy - w^3),$$

for which I do not know if the conjecture is true or not.

If one wants to make a counterexample to the conjecture (0.1), the following lemma will be useful.

**Lemma (3.1).** The following two conditions are equivalent for an $m$-primary ideal $I$.

(a) $\delta_{R}^{2}(R/I) > 0$.

(b) There is an exact sequence

$$0 \longrightarrow R \overset{j}{\longrightarrow} L \overset{p}{\longrightarrow} I \longrightarrow 0$$

with $p \otimes_R k$ an isomorphism (or equivalently $j(1) \in mL$).

**Proof.** (a) $\implies$ (b) Since $M := \Omega_{R}^{2}(R/I)$ is a maximal Cohen-Macaulay module over $R$, the condition (a) says exactly that $M$ contains a free module as a direct summand. Thus there is an epimorphism $\rho : M \longrightarrow R$ and we have a push-out diagram

$$0 \longrightarrow M \longrightarrow F \overset{\pi}{\longrightarrow} I \longrightarrow 0$$

$$\rho \downarrow \quad \phi \downarrow \quad = \downarrow$$

$$0 \longrightarrow R \overset{j}{\longrightarrow} L \overset{p}{\longrightarrow} I \longrightarrow 0,$$

where $\pi$ is a minimal free cover of $I$. Since $\pi = p \cdot \phi$, we see that $(p \otimes k) \cdot (\phi \otimes k) = \pi \otimes k$ is an isomorphism. Note from the diagram that $\phi \otimes k$ is an epimorphism and we have that $p \otimes k$ is an isomorphism.
$(b) \implies (a)$ Let $\phi : F \to L$ be a minimal free cover of $L$. Then we have the commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & X := \text{Ker}(p \cdot \phi) & \longrightarrow & F & \overset{p\phi}{\longrightarrow} & I & \longrightarrow & 0 \\
\psi & & \phi & & = & & \\
0 & \longrightarrow & R & \overset{j}{\longrightarrow} & L & \overset{p}{\longrightarrow} & I & \longrightarrow & 0
\end{array}
\]

Since $(p \cdot \phi) \otimes k = (p \otimes k) \cdot (\phi \otimes k)$ is an isomorphism, we can see that $p \cdot \phi$ is a minimal free cover of $I$. Thus we have $X \cong \Omega^2_R(R/I)$. On the other hand, it follows from the diagram that $\psi$ is an epimorphism and that $\Omega^2_R(R/I)$ has $R$ as a direct summand.

As one of the consequences of (3.1), we get the following

**Proposition (3.2).** Suppose that there exist an integer $r$ and a system of parameters $\{x, y\}$ of $R$ which satisfy the following conditions:

(a) $(x, y) \subseteq m^{r+1},$

(b) $(x, y)m^r = m^{2r+1}.$

Then we have $\delta^2_R(R/m^{2r+1}) > 0.$

**Proof.** Let $I = (x, y)R$ and take the free resolution of $I$:

\[
\begin{array}{cccccc}
0 & \longrightarrow & R & \overset{f_2}{\longrightarrow} & R^2 & \overset{f_3}{\longrightarrow} & I & \longrightarrow & 0
\end{array}
\]

which is part of the Koszul complex. Then we see that $\text{Im}(f_2) \subseteq IR^2 \subseteq m^{r+1}R^2$. Thus we have the exact sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & R & \overset{f_2}{\longrightarrow} & m^rR^2 & \overset{f_3'}{\longrightarrow} & m^rI & \longrightarrow & 0
\end{array}
\]

where $f_3'$ is the restriction to $f_3$ on $m^rR^2$. Since $f_2(1) \in m(m^rR^2)$, it follows from (3.1) that $\delta^2_R(R/m^rI) > 0.$

If $R$ is a regular local ring, then the regular system of parameters satisfies the conditions in (3.2) for $r = 0$. For a non-regular ring $R$, if there is a system of parameters satisfying the conditions $(a), (b)$ in (3.4), then we must have $\text{depth} G = 0$ by (2.2). We do not know if there is a non-regular Gorenstein local ring with the conditions in (3.4), or not.

**References**


