OSCILLATORY SINGULAR INTEGRALS
ON $L^p$ AND HARDY SPACES

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Abstract. We consider boundedness properties of oscillatory singular integrals on $L^p$ and Hardy spaces. By constructing a phase function, we prove that $H^1$ boundedness may fail while $L^p$ boundedness holds for all $p \in (1, \infty)$. This shows that the $L^p$ theory and $H^1$ theory for such operators are fundamentally different.

§1. Introduction

Let $\Phi$ be a real-valued $C^1$ function on $[-1,1]$, $\Phi(0) = \Phi'(0) = 0$, $\lambda \in \mathbb{R}$, and define $T_\lambda$ by

$$T_\lambda f(x) = \text{p.v.} \int_{|x-y| \leq 1} e^{i\lambda \Phi(x-y)} \frac{f(y)}{x-y} f(y) dy. \quad (1)$$

Such operators are called oscillatory singular integral operators and have been studied extensively ([2], [3], [6], [9], [10], [12]). If $\Phi$ is sufficiently smooth and $\Phi^{(k)}(0) \neq 0$ for some $k > 1$, then the $L^p$ and $H^1$ boundedness of $T_\lambda$ is well-known ([13], [6], [7]).

Theorem A. Suppose $\Phi$ is sufficiently smooth and $\Phi^{(k)}(0) \neq 0$ for some $k > 1$; then $T_\lambda$ are uniformly bounded on $L^p(\mathbb{R})$ ($1 < p < \infty$) and $H^1(\mathbb{R})$.

The space $H^1(\mathbb{R})$ is the usual Hardy space $H^1$.

Both the $L^p$ and $H^1$ uniform boundedness of $\{T_\lambda\}_{\lambda \in \mathbb{R}}$ may fail if $\Phi^{(n)}(0) = 0$ for all $n$ ([5], [7]). On the other hand, the condition that $\Phi^{(k)}(0) \neq 0$ for some $k > 1$ is not a necessary condition. By using results of Nagel et al. ([4]) on Hilbert transforms along curves, one can obtain the following $L^2$ result:

Theorem B. Suppose $\Phi$ is even, $\Phi(0) = \Phi'(0) = 0$ and $\Phi'' > 0$. Then the operators $T_\lambda$ are uniformly bounded on $L^2(\mathbb{R})$ if and only if there is a $C > 0$ such that

$$\Phi'(Ct) > 2\Phi'(t), \quad (2)$$

for every $t > 0$.

More recent results due to Carlsson et al. ([1]) imply that the uniform $L^p$ boundedness of $T_\lambda$ holds under exactly the same condition.
Theorem C. Suppose \( \Phi \) is given as in Theorem B and \( p \in (1, \infty) \). Then \( T_\lambda \) are uniformly bounded on \( L^p \) if and only if \( \Phi \) satisfies (2).

Results concerning odd \( \Phi \) can be found in [4] and [1]. For the uniform \( H^1 \) boundedness of \( T_\lambda \) we have the following ([8]):

Theorem D. Suppose \( \Phi(0) = \Phi'(0) = 0 \) and
\[
\Phi''(t) \geq 0,
\]
for \( t > 0 \). Then \( T_\lambda \) are uniformly bounded on \( H^1(\mathbb{R}) \).

To give an example of \( \Phi \) which has vanishing derivatives at \( t = 0 \) but satisfies (2) and (3), we cite the function \( \Phi(t) = e^{-1/t^2} \). We point out that condition (3) is strictly stronger than condition (2). By using interpolation and Theorem B, it is easy to see that (2) is a necessary condition for the uniform \( H^1 \) boundedness of \( T_\lambda \) (when \( \Phi \) is even and convex). In light of Theorem C, it seems reasonable to speculate that (2) may also be a sufficient condition for the uniform \( H^1 \) boundedness of \( T_\lambda \). This turns out to be false. The purpose of this paper is to construct an even, convex function \( \Phi \) which satisfies (2) such that the corresponding \( T_\lambda \)'s are not uniformly bounded on \( H^1(\mathbb{R}) \).

Theorem E. There exists a function \( \Phi \) which is even and satisfies (i) \( \Phi(0) = \Phi'(0) = 0 \) and (ii) \( \Phi'' > 0 \) such that
\[
\sup_{\lambda \in \mathbb{R}} \|T_\lambda\|_{L^p \to L^p} < \infty,
\]
for \( 1 < p < \infty \), and
\[
\sup_{\lambda \in \mathbb{R}} \|T_\lambda\|_{H^1 \to H^1} = \infty.
\]

§2. Construction of \( \Phi \)

For \( k \leq -1 \), we choose a function \( P_k(t) \) on \([2^{k+1} - 2^k/8, 2^{k+1}] \) such that
(i) \( P_k(2^{k+1} - 2^k/8) = 2^{2k+7} \cdot 2^k \cdot 2^{5^2 - 2} \), \( P_k(2^{k+1}) = 2^{2k+2} \);
(ii) \( P_k'(2^{k+1} - 2^k/8) = 24^k + 1 \), \( P_k'(2^{k+1}) = 24^k + 5 \);
(iii) \( 0 < P_k''(t) < C2^k \), for \( t \in [2^{k+1} - 2^k/8, 2^{k+1}] \), where \( C \) is some positive constant.

Define \( g \) on \([0, 1]\) by \( g(0) = 0 \) and \( g(t) = 2^{2k+2^4-k+1} - 2^k \) if \( t \in [2^k, 2^{k+1} - 2^k/8] \);
\( g(t) = P_k(t) \) if \( t \in [2^{k+1} - 2^k/8, 2^{k+1}] \). Clearly we have \( g \in C^1([0, 1]) \). Define \( \Phi(t) \) for \( t \geq 0 \) by
\[
\Phi(t) = \int_0^t g(s)ds.
\]
For \( t \leq 0 \) we let \( \Phi(t) = \Phi(-t) \). Therefore we have \( \Phi \in C^2([1, 1]) \), \( \Phi(0) = \Phi'(0) = 0 \), and \( \Phi''(t) > 0 \) for \( t \neq 0 \). Let \( T_\lambda \) be given by (1). The following result can be found in [14].

Lemma 1. Let \( \{n_j\} \) be a sequence of positive integers such that \( n_{j+1}/n_j \geq \sigma > 1 \) for \( j = 1, 2, \ldots \), and let \( \{\xi_j, \gamma_j\} \) be a sequence of pairs of real numbers. If \( \sum_{j=1}^\infty (\xi_j^2 + \gamma_j^2) < \infty \), then there is a function \( f \in C([0, 2\pi]) \) such that
\[
\int_0^{2\pi} f(t) \cos n_j t dt = \xi_j,
\]
and
\[ \int_0^{2\pi} f(t) \sin nt \, dt = \gamma_j, \]
for \( j = 1, 2, \ldots \).

By Lemma 1, there exists a function \( \eta \in C([0, 2\pi]) \) such that
\[ \int_0^{2\pi} \cos(2^j t) \eta(t) \, dt = 0; \]
and
\[ \int_0^{2\pi} \sin(2^j t) \eta(t) \, dt = 1/j, \]
for \( j = 1, 2, \ldots \).

Define \( b(t) \) on \( \mathbb{R} \) by
\[ b(t) = \eta(t) \text{ if } t \in [0, 2\pi], \quad b(t) = -\eta(-t) \text{ if } t \in [-2\pi, 0), \]
and \( b(t) \equiv 0 \) if \( t \not\in [-2\pi, 2\pi] \). We have \( b \in H^1(\mathbb{R}) \) and \( \|b\|_{H^1} = c_0 > 0 \).

Proposition 1. Let \( \Phi \) be defined as above. Then for \( p \in (1, \infty) \), there exists \( C_p > 0 \) such that
\[ \sup_{\lambda \in \mathbb{R}} \|T_\lambda\|_{L^p \to L^p} \leq C_p. \]

Proof. Since \( 2^{2k} \leq \Phi'(t) \leq 2^{2k+2} \) for \( t \in [2^k, 2^{k+1}] \), we have
\[ \Phi'(8t) > 2\Phi'(t) \]
for \( t \geq 0 \). Therefore (8) follows from Theorem C.

Proof. Since \( 2^{2k} \leq \Phi'(t) \leq 2^{2k+2} \) for \( t \in [2^k, 2^{k+1}] \), we have
\[ \Phi'(8t) > 2\Phi'(t) \]
for \( t \geq 0 \). Therefore (8) follows from Theorem C. \( \square \)

Proposition 2. Let \( a(x) \) be a function defined on \( [-\delta, \delta] \) and satisfying \( \|a\|_\infty \leq (2\delta)^{-1} \) and
\[ \int_{-\delta}^{\delta} a(x) \, dx = 0. \]

Then
\[ \int_{|x| > 2\delta} \left| \int_{-\delta}^{\delta} e^{i\lambda \Phi(x-y)} a(y) \, dy \right| \, dx \leq \|T_\lambda a\|_1 + C, \]
for some \( C \) which is independent of \( a \) and \( \lambda \).

Proof.
\[ \|T_\lambda a\|_1 \geq -\int_{|x| \leq 2\delta} |T_\lambda a(x)| \, dx \]
\[ -\int_{|x| > 2\delta} \int_{-\delta}^{\delta} \left| \frac{1}{x} - \frac{1}{x-y} \right| |a(y)| \, dy \, dx + \int_{|x| > 2\delta} \frac{1}{|x|} \left| \int_{-\delta}^{\delta} e^{i\lambda \Phi(x-y)} a(y) \, dy \right| \, dx \]
\[ = -I_1 - I_2 + I_3. \]
By Proposition 1 and Hölder’s inequality, we have
\[ I_3 \leq C\delta^{1/2} \|T_\lambda a\|_2 \leq C. \]
For $I_2$ we have

$$I_2 \leq \int_{2\delta}^{\infty} \frac{\delta^2 \|a\|_\infty}{x^2} dx \leq C.$$  

Therefore, (9) holds. \hfill \square

We now prove Theorem E. Let $N$ be a large integer and $\lambda = 2^{N+5}\pi$. Define $a_N(x)$ by

$$a_N(x) = (2^{N/3+\pi})b(2^{N/3+\pi}x).$$

Therefore we have $\text{supp}(a_N) \subset [-2^{-N/3-3}, 2^{-N/3-3}]$, and $\|a_N\|_{H^1} \geq c_0 > 0$ for some $c_0$ independent of $N$.

For $k \leq -1$, $t \in [2^k, 2^{k+1} - 2^k/8]$ we have

$$\Phi(t) = \Phi(2^k) + 2^{2k}(t - 2^k) + 2^{4k}(t - 2^k)^2.$$  

Therefore, for $-N/3 \leq k \leq -1$,

$$\int_{2^k+2^k/8 \leq x \leq 2^{k+1}-2^k/4} \frac{1}{x} \left| \int_{\mathbb{R}} e^{i\lambda \Phi(x-y)} a_N(y) dy \right| dx$$

$$\geq \int_{2^k+2^k/8 \leq x \leq 2^{k+1}-2^k/4} \frac{1}{x} \left| \int_{\mathbb{R}} e^{i\lambda(\Phi(2^k)+2^{2k}(x-y-2^k))} a_N(y) dy \right| dx - C\lambda \cdot 2^{6k}$$

$$= \ln(14/9) \left| \int_{\mathbb{R}} e^{-i\lambda \cdot 2^k y} a_N(y) dy \right| - C\lambda \cdot 2^{6k}$$

$$= 2 \ln(14/9)(2N/3 + 2k + 1)^{-1} - C \cdot 2^{N+6k},$$

where we used (7). Hence we obtain

$$\int_{|x| \geq 2^{-N/3}} \frac{1}{|x|} \left| \int_{\mathbb{R}} e^{i\lambda \Phi(x-y)} a_N(y) dy \right| dx$$

$$\geq 2 \ln(14/9) \sum_{-N/3 \leq k \leq -N/4} (2N/3 + 2k + 1)^{-1} - C \sum_{k \leq -N/4} 2^{N+6k}$$

$$\geq c_1 \ln N,$$

where $c_1$ is some absolute constant. By Proposition 2, we have

$$\|T_\lambda a_N\|_{H^1} \geq \|T_\lambda a_N\|_{L^1} \geq c'_1 \ln N \|a_N\|_{H^1},$$

for $\lambda = 2^{N+5}\pi$. Thus we have

$$\sup_{\lambda \in \mathbb{R}} \|T_\lambda\|_{H^1 \to H^1} = \infty.$$

\hfill \square
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REFERENCES


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