SOME REMARKS ON THE OPERATOR
OF FOIAS AND WILLIAMS

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ABSTRACT. In this paper we study the Foias-Williams operator

$$T(H_g) = \begin{pmatrix} S^* & H_g \\ 0 & S \end{pmatrix}$$

where $g \in L^\infty$, and $H_g$ is a Hankel operator with symbol $g$. We exhibit a relationship between the similarity of $T(H_g)$ to a contraction and the rate of decay of $\{|g_n|\}_{n=0}^\infty$, the absolute values of the Fourier coefficients of the symbol $g$.

Let $\mathcal{H}$ denote a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. Recall that an operator $T$ in $\mathcal{L}(\mathcal{H})$ is said to be \textit{polynomially bounded} if there exists an $M \geq 1$ such that

$$\|p(T)\| \leq M \sup \{|p(\zeta)| : |\zeta| = 1\}$$

for all polynomials $p$. We denote the class of all polynomially bounded operators by $(PB)$. Also, an operator $T$ is similar to a contraction (notation: $T \in (SC)$) if there exists a bounded invertible operator $L$ such that $\|LTL^{-1}\| \leq 1$. Halmos [4] raised the question whether every polynomially bounded operator is similar to a contraction; this question is still open.

While there is a number of results dealing with sufficient conditions for a polynomially bounded operator to be similar to a contraction (cf. [9], [6], [5]), there are very few publications dedicated to the search for a counterexample. In [3] (see also [2]) Foias and Williams have studied the operators of the form

$$T(X) = \begin{pmatrix} S^* & X \\ 0 & S \end{pmatrix}$$

acting on $H^2 \oplus H^2$, the direct orthogonal sum of two copies of the Hardy space $H^2$ (to be defined below). Here $S$ is a forward unilateral shift on $H^2$ and $X \in \mathcal{L}(H^2)$. In particular, they conjectured that there exists a Hankel operator $H_g$ with symbol $g$ (to be defined below) such that the operator $T(H_g) \in (PB) \setminus (SC)$.

In this paper we continue this line of investigation and we show that the membership in the aforementioned classes depends on the rate at which the sequence $\{|g_n|\}_{n \in \mathbb{N}_0}$ of the Fourier coefficients of $g$ tends to 0.

Before stating the main results, however, we briefly introduce some notation and terminology. As usual $\mathbb{N}$ [resp., $\mathbb{N}_0$] denotes the set of positive [resp., non-negative]
integers, \( \mathbb{C} \) is the complex plane and \( T \) is the unit circle in \( \mathbb{C} \). \( L^\infty \) is the algebra of all bounded Lebesgue-measurable functions on \( T \), and \( H^\infty \) is its subalgebra consisting of those functions whose negative Fourier coefficients vanish. Similarly, \( H^2 \) is the space of square-integrable functions on \( T \) whose negative Fourier coefficients vanish. Finally, \( \ell^2 \) is the Hilbert space of all sequences \((x_n)_{n \in \mathbb{N}_0}\) such that \( \sum_{n \in \mathbb{N}_0} |x_n|^2 < \infty \)

and \( \|(x_n)\|_{\ell^2} = \left\{ \sum_{n \in \mathbb{N}_0} |x_n|^2 \right\}^{1/2} \). Furthermore, \( z^n = e^{int}, n \in \mathbb{N}_0 \), is the standard orthonormal basis of \( H^2 \), \( e_n, n \in \mathbb{N}_0 \), is the standard basis of \( \ell^2 \) (\( e_n \) is a sequence whose only non-zero term is in the \( n \)-th position), and there exists a Hilbert space isomorphism \( V : H^2 \to \ell^2 \) such that \( Vz^n = e_n, n \in \mathbb{N}_0 \). In what follows we shall identify \( H^2 \) with \( \ell^2 \) (and \( z^n \) with \( e_n \)) without further comment. If \( g \in L^\infty, H_g \) is the Hankel operator with symbol \( g \), i.e., \( H_g \) is a bounded linear operator on \( H^2 \) satisfying \( H_g(z^n) = S^*g \), in this situation, using the aforementioned identification between \( H^2 \) and \( \ell^2 \), \( H_g \) is represented by the matrix \((g_{i+j})_{i,j=0}^\infty \), where \( g_n \) denotes the \( n \)-th Fourier coefficient of \( g \).

**Lemma 1.** Suppose \( g \in H^\infty \) and the series \( \sum_{n=1}^\infty n^2|g_n|^2 \) does not converge. Then \( T(H_g) \) is not similar to a contraction. In particular, this happens when \( |g_n| = (n+1)^{-3/2} \).

**Proof.** By [2, Theorem 5.3], \( T(X) \in (SC) \) if and only if \( \exists f \in H^2 \) such that the operator \( W \) defined on polynomials by

\[
Wz^n = S^*(A_{n+1} \cdot 1 - S^n f), \quad n \in \mathbb{N}_0,
\]

extends to a bounded linear operator on \( H^2 \). Here

\[
A_n = \sum_{j=0}^{n-1} S^{*(n-1-j)} X S^j, \quad n \in \mathbb{N}.
\]

In particular, when \( X = H_g \), the Hankel operator with symbol \( g \in L^\infty(T) \),

\[
A_n = \sum_{j=0}^{n-1} S^{*(n-1-j)} H_g S^j = \sum_{j=0}^{n-1} S^{*n-1} H_g = n S^{*n-1} H_g,
\]

and therefore

\[
Wz^n = S^*((n+1)S^n H_g \cdot 1 - S^n f) = (n+1)S^{n+1} H_g \cdot 1 - S^{n+1} f, \quad n \in \mathbb{N}_0.
\]

Since \( H_g 1 = g \) we have that \( Wz^n = (n+1)S^{n+1} g - S^{n+1} f \). Clearly, such an operator \( W \), in the basis \( \{z^n\}_{n \in \mathbb{N}_0} \) is represented by a matrix

\[
\left( (Wz^r, z^s) \right)_{r,s=0}^\infty
\]

and it is easy to see that

\[
\langle Wz^r, z^s \rangle = (r+1)\langle g, z^{r+s+1} \rangle - \langle f, z^{r+s+1} \rangle = (r+1)g_{r+s+1} - f_{r+s+1}.
\]

In other words, \( W \) is represented by a matrix

\[
(1) \quad A = \left( (r+1)g_{r+s+1} - f_{r+s+1} \right)_{r,s=0}^\infty,
\]
and for \( W \) to be bounded on \( H^2 \) it is necessary that the columns of \( A \) be square summable, i.e.,

\[
\sum_r |(r+1)g_{r+s+1} - f_{r+s+1}|^2 < \infty, \quad \forall s \in \mathbb{N}_0.
\]

In particular we have that \( \sum_r |(r+1)g_r - f_r|^2 < \infty \), which in view of \( f \in H^2 \) implies that \( \sum_r r^2|g_r|^2 < \infty \), and the lemma is proved.

The main result of this paper is that \(-\frac{3}{2}\) is indeed a limiting point. More precisely, we have the following

**Theorem 2.** Suppose that \( \alpha > \frac{3}{2} \). Then there exists a function \( g \in H^\infty \) such that

\[
|g_n| = \frac{1}{(n+1)\alpha}, \quad n \in \mathbb{N}_0 \quad \text{and such that, with } f = 0, \text{ the matrix (1) is bounded on } \ell^2.
\]

Before we can prove Theorem 2 we need to recall a few results. First, one knows that there exists a sequence \( \{\epsilon_n\}_{n \in \mathbb{N}_0} \subset \{-1, 1\} \) and a constant \( C \) such that

\[
\sup_{z \in \mathbb{T}} \left| \sum_{n=0}^{N} \epsilon_n z^n \right| \leq C\sqrt{N}, \quad \forall N \in \mathbb{N}.
\]

Since the numbers \( \epsilon_n \) were independently introduced by Shapiro [8] and Rudin [7], they are generally referred to as the Rudin-Shapiro coefficients.

Next, we recall that for two matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \) the Schur product of \( A \) and \( B \) is defined as \( A \ast B = (a_{ij}b_{ij}) \). A matrix \( M \) is said to be a Schur multiplier if \( M \ast X \) defines a bounded operator on \( \ell^2 \) for every \( X \in \mathcal{L}(\ell^2) \). The following useful test comes from [1]:

**Theorem 3.** Let \( M \) be a matrix with \( \lim_{k} m_{jk} = 0 = \lim_{j} m_{jk} \), and suppose that

\[
\sum_{j,k=1}^{\infty} |m_{j,k} - m_{j,k+1} - m_{j+1,k} + m_{j+1,k+1}| < \infty.
\]

Then \( M \) is a Schur multiplier.

**Proof of Theorem 2.** Given \( \alpha > \frac{3}{2} \), we define \( g_n = \frac{\epsilon_n}{(n+1)^\alpha}, \quad n \in \mathbb{N}_0 \), where \( \epsilon_n \) denotes the \( n \)th Rudin-Shapiro coefficient. In this situation, with \( f = 0 \), the matrix (1) becomes

\[
A = \left( (r+1)\frac{\epsilon_{r+s}}{(r+s+1)^\alpha} \right)_{r,s=0}^{\infty}.
\]

Since \( \alpha > \frac{3}{2} \), \( \exists \beta > 0 \) such that \( \alpha = 2\beta + \frac{3}{2} \). Using the Schur product we can rewrite \( A = M \ast B \) where

\[
M = \left( \frac{r+1}{(r+s+1)^{1+\beta}} \right), \quad B = \left( \frac{\epsilon_{r+s}}{(r+s+1)^{\frac{1}{2}+\beta}} \right).
\]
First we show that $B$ defines a bounded linear operator on $\ell^2$. We notice that $B$ is the Hankel operator corresponding to the function $h(z) = \sum_{k=0}^{\infty} \epsilon_k (k+1)^{(1/2) + \beta} z^k$.

Therefore it suffices to show that $h \in H^\infty$ because in that case $\|B\| \leq \|h\|_{\infty}$.

An easy application of the Abel summation by parts allows us to rewrite

$$h(z) = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} \epsilon_j z^j \right) \left[ \frac{1}{(k+1)^{(1/2) + \beta}} - \frac{1}{(k+2)^{(1/2) + \beta}} \right]$$

and therefore

$$\|h\|_{\infty} \leq \sup_{z \in T} \sum_{k=0}^{\infty} \left| \sum_{j=0}^{k} \epsilon_j z^j \right| \left( (k+2)^{(1/2) + \beta} - (k+1)^{(1/2) + \beta} \right) \frac{(k+1)^{(1/2) + \beta} - (k+2)^{(1/2) + \beta}}{(k+1)^{(1/2) + \beta} (k+2)^{(1/2) + \beta}}.$$

By the mean value theorem

$$(k+2)^{(1/2) + \beta} - (k+1)^{(1/2) + \beta} = \left( \frac{1}{2} + \beta \right) (k+1 + \theta)^{(1/2) + \beta - 1}$$

for some $0 < \theta < 1$. Thus, employing (2),

$$\|h\|_{\infty} \leq \sum_{k=0}^{\infty} C \sqrt{k} \left( \frac{1}{2} + \beta \right) \frac{(k+1 + \theta)^{(1/2) + \beta - 1}}{(k+1)^{(1/2) + \beta} (k+2)^{(1/2) + \beta}}.$$

and it is easy to see that this series converges, from which it follows that $B$ is a bounded operator.

Finally, we show that $M$ is a Schur multiplier. We apply Theorem 3 with $m_{jk} = \frac{j+1}{(j+k+1)^{1+\beta}}$. Clearly, both limits are 0. Therefore, it remains to show the convergence of the double series

$$\sum_{j,k=1}^{\infty} \left| \frac{j+1}{(j+k+1)^{1+\beta}} - \frac{j+1}{(j+k+2)^{1+\beta}} - \frac{j+2}{(j+k+2)^{1+\beta}} + \frac{j+2}{(j+k+3)^{1+\beta}} \right|.$$

One knows that if a function $f(x,y)$ has continuous partial derivatives of the second order, then

$$(3) \quad f(x,y) - f(x,y+1) - f(x+1,y) + f(x+1,y+1) = \frac{\partial^2 f}{\partial x \partial y}(x+\theta, y+\eta),$$

for some $0 < \theta, \eta < 1$. Using (3) with $f(x,y) = \frac{x+1}{(x+y+1)^{1+\beta}}$, we obtain that the series above can be written as

$$(\beta + 1) \sum_{j,k=1}^{\infty} \left| \frac{(\beta + 1)(j+\theta+1) - (k+\eta)}{(j+k+\theta+\eta+1)^{3+3}} \right|.$$

Since this double series is obviously convergent, the theorem is proved. □

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