GEOMETRIC INDICES
AND THE ALEXANDER POLYNOMIAL OF A KNOT

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Abstract. It is well-known that any Laurent polynomial \( \Delta(t) \) satisfying \( \Delta(t) = \Delta(t^{-1}) \) and \( \Delta(1) = \pm 1 \) is the Alexander polynomial of a knot in \( S^3 \). We show that \( \Delta(t) \) can be realized by a knot which has the following properties simultaneously: (i) tunnel number 1; (ii) bridge index 3; and (iii) unknotting number 1.

1. Introduction

In this paper, we study some geometric indices of a knot in the 3-sphere \( S^3 \): the tunnel number, the unknotting number, the bridge index, and the Alexander polynomial. The tunnel number of a knot \( K, \tau(K) \), is the minimum number of mutually disjoint arcs properly embedded in the exterior of \( K \) such that the complementary space of \( K \cup \{ \text{arcs} \} \) is a handlebody. The unknotting number of \( K, u(K) \), is the minimum number of exchanges of crossings required to deform \( K \) into a trivial knot over all knot diagrams representing \( K \). The bridge index of \( K, b(K) \), is the minimum number of \( n \) such that \( (S^3, K) \) is a tangle sum \( (B^3_1, t_1) \cup (B^3_2, t_2) \), where \( (B^3_i, t_i) \) is a trivial \( n \)-string tangle, \( i = 1, 2 \). We denote by \( \Delta_K(t) \) the Alexander polynomial of \( K \).

Theorem 1. For any Laurent polynomial \( A(t) \in \mathbb{Z}[t, t^{-1}] \) with \( A(1) = \pm 1 \) and \( A(t) = A(t^{-1}) \), where \( \pm \) means "equal up to units", there exists a knot \( K \) in \( S^3 \) which satisfies the following conditions simultaneously:

\[
\begin{align*}
(i) & \quad \Delta_K(t) = A(t), \\
(ii) & \quad \tau(K) = 1, \\
(iii) & \quad b(K) = 3, \text{ and} \\
(iv) & \quad u(K) = 1.
\end{align*}
\]

Theorem 2. There exists a set of mutually inequivalent knots \( \{ K_i \}_{i \in \mathbb{Z}} \) such that each \( K_i \) satisfies the following conditions simultaneously:

\[
\begin{align*}
(i) & \quad \Delta_K(t) = 1, \\
(ii) & \quad \tau(K) = 1, \\
(iii) & \quad b(K) = 3, \text{ and} \\
(iv) & \quad u(K) = 1.
\end{align*}
\]
Note that a tunnel number one knot is strongly invertible [1]. For any Laurent polynomial $A(t)$ with $A(1) = \pm 1$ and $A(t) = A(t^{-1})$, Kondo [8] and Sakai [17] constructed an unknotted number one knot whose Alexander polynomial is equal to $A(t)$. And then Sakai [18] showed that his knots are strongly invertible.

Let $F_g$ be a genus $g$ surface standardly embedded in $S^3$. Any knot can be embedded in $F_g$ for some $g$. A torus knot is a knot which is embedded in $F_1$. According to the observation of C. Morin and M. Saito; cf. [13, p.138], a tunnel number one knot can be embedded in either $F_1$ or $F_2$.

From Theorem 1, we get the following result.

**Corollary.** For any Laurent polynomial $A(t)$ with $A(1) = \pm 1$ and $A(t) = A(t^{-1})$, there exists a knot embedded in $F_2$ whose Alexander polynomial is equal to $A(t)$.

The Alexander polynomial of a torus knot has some restricted form. This is also true for the Alexander polynomial of a 2-bridge knot (cf. Hartley [5]). Sakuma proposed the following conjecture.

**Conjecture.** For any Laurent polynomial $A(t)$ with $A(1) = \pm 1$ and $A(t) = A(t^{-1})$, there exists a tunnel number one knot (resp. a 3-bridge knot) whose Alexander polynomial is equal to $A(t)$.

Theorem 1 implies that this conjecture is true. The proofs are given in Sections 2 and 3; they are obtained by a small alteration of Sakai’s construction in [17].

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2. **Geometric indices**

To prove Theorems 1 and 2, we consider two classes of knots $K(\alpha_1, \cdots, \alpha_n)$ and $J(\alpha, \beta, \gamma)$ as illustrated in Figure 1.
Here \( T_{\alpha} \) is a 2-string tangle with left hand \( \alpha \)-full twists, and \( U_{\alpha} \) is a 4-string tangle with left hand \((2\alpha+1)\)-half twists of two bands; see Figure 2. Note that “the band parts” of \( U_{\alpha} \) are untwisted.

Then we easily see the upper bounds of the geometric indices.

**Lemma 1.** The unknotting numbers of \( K(\alpha_1, \cdots, \alpha_n) \) and \( J(\alpha, \beta, \gamma) \) are less than or equal to one.

**Proof.** Exchange the crossings * in Figures 1a and 1b, and we get a trivial knot. \( \square \)

**Lemma 2.** The tunnel numbers of \( K(\alpha_1, \cdots, \alpha_n) \) and \( J(\alpha, \beta, \gamma) \) are less than or equal to one.

**Proof.** Let \( K = K(\alpha_1, \cdots, \alpha_n) \). For simplicity, we prove the lemma only in the case of \( n = 4 \). The proof in the general case is similar. We add the arc \( \tau \) in \( E(K) \) as illustrated in Figure 3a, and we have the deformations indicated in Figures 3a–3d. Then \( \tau_2 \) in Figure 3d is an unknotting tunnel of the 2-bridge knot; see [1]. Hence \( \tau(K) \leq 1 \).

For \( J(\alpha, \beta, \gamma) \), the unknotting tunnel \( \tau \) and the deformation are given in Figures 4a–4c. \( \square \)

To state Lemma 3, we recall the following results.

**Proposition 1** ([12]). Let \( K \) be a knot, and \( \Sigma_2(K) \) the two-fold branched covering space of \( S^3 \) branched over \( K \). If the unknotting number of \( K \) is one, then there exists a strongly invertible knot \( K' \) such that \( \Sigma_2(K) \) can be obtained by the Dehn surgery of type \( p/2 \) on \( K' \), where \( p \) is an odd integer,

**Proposition 2** (Cyclic Surgery Theorem [3]). Let \( K \) be a knot which is not a torus knot, and \( K(r) \) the 3-manifold obtained by the Dehn surgery of type \( r \) on \( K \) \((r \in \mathbb{Q})\). If the fundamental group \( \pi_1(K(r)) \) is cyclic, then \( r \) is an integer.

We denote by \( C(\alpha_1, \cdots, \alpha_n) \) \((\alpha_i \in \mathbb{Z})\) the 2-bridge knot as shown in Figure 5; see [2].

![Figure 2](image-url)
Proposition 3 ([6]). We have the transformations of 2-bridge knots as follows:

\[
C(x_1, \cdots, x_n, a, y, y_1, \cdots, y_m) \cong C(x_1, \cdots, x_n + \epsilon, ||a||, (-1)^a(y + \epsilon), (-1)^ay_1, \cdots, (-1)^ay_m),
\]

where \(\epsilon = a/||a||\) and \(||a|| = \left\lfloor \frac{2\epsilon}{2\epsilon + \epsilon_4} \right\rfloor\) with \(\epsilon_4 = \frac{a}{||a||}\).

Lemma 3. Let \(K\) be one of \(K(\alpha_1, \cdots, \alpha_n)\) \((n \geq 3\) and \(\alpha_n \neq 0\)) and \(J(\alpha, \beta, \gamma)\) \((\beta \neq 0\) or \(\gamma \neq 0)\). Then \(b(K) = 3\).

Proof. Case 1. Let \(K = K(\alpha_1, \cdots, \alpha_n)\) \((n \geq 3\) and \(\alpha_n \neq 0)\). It is easy to see that \(b(K) \leq 3\). We show that \(b(K) \geq 3\). If \(K\) is a 2-bridge knot, then the two-fold branched covering space of \(S^3\) branched over \(K\), denoted by \(\Sigma_2(K)\), is a lens space. So \(\pi_1(\Sigma_2(K))\) is cyclic.

From Proposition 1 and the construction given in [12], we have that \(\Sigma_2(K)\) is a 3-manifold obtained by Dehn surgery of type \(p/2\) on the 2-bridge knot \(C = C(-1, -4a_2, -1, -4a_3, -1, \cdots, -4a_n)\), where \(p\) is an odd integer; see Figure 6 in Section 3. By Proposition 2, it is sufficient to prove that \(C\) is not a torus knot. If \(C\) is a torus knot, then \(C \cong \pm C(2, -2, \cdots, 2, -2)\) which is a torus knot of type \((2, m)\). Using Proposition 3, we deform \(C\) as follows:

\[
\begin{aligned}
&\cong C(-1, -4a_2, -1, -4a_3, -1, -4a_4, -1, \cdots, -4a_n) \\
&\cong C(-1 + \epsilon_2, ||-4a_2||, -1 + \epsilon_2, -4a_3, -1, -4a_4, -1, \cdots, -4a_n) \\
&\cong C(0, -2, \cdots, -2, 0, -4a_3, -1, -4a_4, -1, \cdots, -4a_n), \quad \text{if } \epsilon_2 = 1; \\
&\cong C(-2, 2, \cdots, -2, -2, -4a_3, -1, -4a_4, -1, \cdots, -4a_n), \quad \text{if } \epsilon_2 = -1 \\
&\cong \begin{cases} 
C(2, -2, \cdots, 2, -2 + 4a_3, 0, -2, \cdots, -2, 0, \cdots, -4a_n), & \text{if } (\epsilon_2, \epsilon_4) = (1, 1); \\
C(2, -2, \cdots, 2, -2 + 4a_3, -2, ||-4a_3||, -2, \cdots, -4a_n), & \text{if } (\epsilon_2, \epsilon_4) = (1, -1); \\
C(2, -2, \cdots, 2, -2 + 4a_3, -2, ||-4a_3||, -2, \cdots, -4a_n), & \text{if } (\epsilon_2, \epsilon_4) = (-1, 1); \\
C(2, -2, \cdots, 2, -2 + 4a_3, -2, ||-4a_3||, -2, \cdots, -4a_n), & \text{if } (\epsilon_2, \epsilon_4) = (-1, -1),
\end{cases}
\end{aligned}
\]

where \(\epsilon_i = -a_i/||a_i||\) and \(||a_i|| = \left\lfloor \frac{2\epsilon_i}{2\epsilon_i + \epsilon_4} \right\rfloor\) with \(\epsilon_4 = \frac{a_i}{||a_i||}\). So \(C\) is not a torus knot.

Case 2. Let \(K = J(\alpha, \beta, \gamma)\), where \(\beta \neq 0\) or \(\gamma \neq 0\).

From Figure 1b, we see that \(b(K) \leq 3\). Next we show that \(b(K) \geq 3\). In the same way as in Case 1, we consider the two-fold branched covering space \(\Sigma_2(K)\)
of $S^3$ branched over $K$ which is obtained by Dehn surgery of type $p/2$ on the 2-bride knot $C = \{1, -4, -1, -4\beta, -1, -4\gamma\}$, where $p$ is an odd integer. Using Proposition 3, we deform $C$ as follows:

\[
\begin{align*}
C(1, -4, -1, -4\beta, -1, -4\gamma) \\
\cong C(0, 2, -2, 2, -2, -4\beta, -1 + \epsilon_\gamma, || - 4\gamma||) \\
\cong C(-2, 2, -2, -4\beta, 1 + \epsilon_\gamma, || - 4\gamma||) \\
\not\equiv \pm C(2, -2, \cdots, 2, -2),
\end{align*}
\]

where $\epsilon_\gamma = -\gamma/|\gamma|$ and $|| - 4\gamma|| = (\underbrace{-2\epsilon_\gamma, 2\epsilon_\gamma, \cdots, (-1)^{2\gamma-1}\epsilon_\gamma}_{4|\gamma|-1}).$

From this, $C$ is not a torus knot. Hence $\pi_1(\Sigma_2(K))$ is non-cyclic, and thus $b(K) \geq 3.$

\[\square\]

3. Alexander polynomial

Now we calculate the Alexander polynomials of the above two classes of knots.

Case 1. $K(\alpha_1, \cdots, \alpha_n)$. We deform $K(\alpha_1, \cdots, \alpha_n)$ as illustrated in Figures 6a–6d. Thus $(S^3, K(\alpha_1, \cdots, \alpha_n))$ is homeomorphic to $(T(1), K)$ in Figures 6e, where $T(1)$ is obtained by performing the Dehn surgery of type $+1$ in $T$ (cf. [17]).

To calculate $\Delta_K(t)$, the Alexander polynomial of $K$, we use the infinite cyclic covering space $\tilde{E}(K)$ of the exterior of $K$, $E(K)$; see Chapter 7 in [16]. The Alexander module of $K$, $H_1(\tilde{E}(K); \mathbb{Z})$, is a $\mathbb{Z}[t, t^{-1}]$-module, where $t$ acts in $\tilde{E}(K)$ as a covering translation. Let $\{\tilde{T}_i\}_{i \in \mathbb{Z}}$ be the system of the lifts of $T$ in $\tilde{E}(K)$, and $c_0$ the surgery coefficient of $\tilde{T}_i$. Then $\tilde{T}_j = t^{\ell} \tilde{T}_0$. We put $c_j = \ell \ell(\tilde{T}_0, \tilde{T}_j) (j \neq 0).$

Then $c_j = c_{-j}$ for $j \geq 1$. Now $\tilde{E}(K)$ is constructed from $D^2 \times R^1$ by removing the regular neighborhoods of $\{\tilde{T}_i\}_{i \in \mathbb{Z}}$ and replacing them with solid tori, where the surgery coefficients are $c_0$. Then we have

\[
\Delta_K(t) = 1 - 2\alpha_1 + \alpha_1(t + t^{-1}) + \alpha_2(t^2 - 2t + 2 - 2t^{-1} + t^{-2}) + \sum_{i=3}^{n} \alpha_i(t^i - 2t^{i-1} + t^{-i-2} + t^{-i+2} - 2t^{-i+1} + t^{-i}).
\]

(4)

Let $(\alpha_1, \cdots, \alpha_n)$ be the solution of

\[
\begin{align*}
c_n &= \alpha_n, \\
c_{n-1} &= \alpha_{n-1} - 2\alpha_n, \\
c_i &= \alpha_i - 2\alpha_{i+1} + \alpha_{i+2} (1 \leq i \leq n - 2), \\
c_0 &= 1 - 2 \sum_{j=1}^{n} c_j.
\end{align*}
\]

Then, for any polynomial $A(t) = \sum_{j=-n}^{n} c_j t^j$ with $A(1) = 1$ and $c_j = c_{-j} (j \geq 1)$, the Alexander polynomial of $K(\alpha_1, \cdots, \alpha_n)$ is equal to $A(t)$.

As an example, we show it for the knot $K(-1, 1, -2)$ given in Figure 7. The infinite cyclic covering space is given in Figure 8.

Case 2. $J(\alpha, \beta, \gamma)$. We calculate the Alexander polynomial of $J(\alpha, \beta, \gamma)$ in the parallel way as in Case 1. $(S^3, J(\alpha, \beta, \gamma))$ is homeomorphic to $(T(1), K)$ in Figure 9c, where $T(1)$ is obtained by performing the Dehn surgery of type $+1$ in $T$. The infinite cyclic covering space of $\tilde{E}(J(\alpha, \beta, \gamma))$ is shown in Figure 10, where $\{T_i\}_{i \in \mathbb{Z}}$
Figure 6

Figure 7
Figure 8

Figure 9

Figure 10
is the system of the lifts of $T$. We see that

(i) $lk(\tilde{T}_0, \tilde{T}_1) = \alpha + 2\beta - 2\gamma - 1$,
(ii) $lk(\tilde{T}_0, \tilde{T}_2) = 1 + \gamma$, and
(iii) $lk(\tilde{T}_0, \tilde{T}_1) = 0$ for $i \geq 3$.

Therefore, we have

$$\Delta_{J(\alpha, \beta, \gamma)}(t) \equiv 2\gamma - 2\alpha - 4\beta + 1 + (\alpha + 2\beta - 2\gamma - 1)(t + t^{-1}) + (\gamma + 1)(t^2 + t^{-2}).$$

If $A(t) = \sum_{i=-2}^{2} c_i t^i$, where $c_i = c_{-i}$ and $A(1) = 1$, then the Alexander polynomial of $J(\alpha, \beta, \gamma)$ is equal to $A(t)$, where $(\alpha, \beta, \gamma) = (c_1 + 2c_2 + 1, -1, c_2 - 1)$. In particular, $\Delta(J(-2\beta - 1, \beta, -1)) = 1$. We denote $J(-2\beta - 1, \beta, -1)$ by $K_\beta$.

To prove Lemma 4 below, we introduce a Laurent polynomial invariant ([7], Theorem 1.1), $c_0(K; x)$, determined by the following:

(i) $c_0(O; x) = 1, x c_0(L_+; x) - c_0(L_-; x) = c_0(L_0; x)$, where $O$ is a trivial knot, $L_+, L_-$ are knots and $L_0$ is a 2-component link, which are identical except near one point where they are as in Figure 11.

(ii) If $L = L_1 \cup L_2$ is a 2-component link with linking number $\lambda$, then

$$c_0(L; x) = (x - 1) x^{-\lambda} c_0(L_1; x) c_0(L_2; x).$$

This polynomial is a version of the first term of the skein polynomial [4], [11], [15].

**Lemma 4.** Each $K_\beta$ ($\beta \in \mathbb{Z}$) is non-trivial and $K_i \not\cong K_j$ if $i \neq j$.

**Proof:** From Figure 12, we calculate the Laurent polynomial invariant of $K_\beta$, $c_0(K_\beta; x)$. From (i), we have

$$xc_0(K_\beta; x) - 1 = (x - 1)x^b c_0(O; x) c_0(C(2, -\beta, -2, -2\beta, -2, 2); x),$$

and so, we have

$$c_0(K_\beta; x) = x^{-1} + (x - 1)x^{-1} c_0(C(2, -\beta, -2, -2\beta, -2, 2); x)$$

$$= x^{-3} - 3x^{-2} + 3x^{-1} + x^\beta - 3x^\beta - 3x^\beta - 3x{\beta} - x^{\beta+1},$$

from which we obtain the result.

**Concluding remark.** From Proposition 1.3 of [14], our two classes of knots have (1, 1)-decompositions. If a knot has a (1, 1)-decomposition, then its tunnel number is less than or equal to 1. The arc $\tau^*$ in Figure 3 is also an unknotted tunnel.

From the results of [9], [19], [20], an unknotted number 1 knot $K$ with $b(K) \leq 3$ is a hyperbolic knot or a trefoil knot. So our knots are hyperbolic knots.
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