GEOMETRIC INDICES
AND THE ALEXANDER POLYNOMIAL OF A KNOT

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Abstract. It is well-known that any Laurent polynomial $\Delta(t)$ satisfying $\Delta(t) = \Delta(t^{-1})$ and $\Delta(1) = \pm 1$ is the Alexander polynomial of a knot in $S^3$. We show that $\Delta(t)$ can be realized by a knot which has the following properties simultaneously: (i) tunnel number 1; (ii) bridge index 3; and (iii) unknotting number 1.

1. Introduction

In this paper, we study some geometric indices of a knot in the 3-sphere $S^3$: the tunnel number, the unknotting number, the bridge index, and the Alexander polynomial. The tunnel number of a knot $K$, $\tau(K)$, is the minimum number of mutually disjoint arcs properly embedded in the exterior of $K$ such that the complementary space of $K \cup \{\text{arcs}\}$ is a handlebody. The unknotting number of $K$, $u(K)$, is the minimum number of exchanges of crossings required to deform $K$ into a trivial knot over all knot diagrams representing $K$. The bridge index of $K$, $b(K)$, is the minimum number of $n$ such that $(S^3, K)$ is a tangle sum $(B^3_1, t_1) \cup (B^3_2, t_2)$, where $(B^3_i, t_i)$ is a trivial $n$-string tangle, $i = 1, 2$. We denote by $\Delta_K(t)$ the Alexander polynomial of $K$.

Theorem 1. For any Laurent polynomial $A(t) \in \mathbb{Z}[t, t^{-1}]$ with $A(1) = \pm 1$ and $A(t) = A(t^{-1})$, where $\equiv$ means “equal up to units”, there exists a knot $K$ in $S^3$ which satisfies the following conditions simultaneously:

(i) $\Delta_K(t) = A(t)$,
(ii) $\tau(K) = 1$,
(iii) $b(K) = 3$, and
(iv) $u(K) = 1$.

Theorem 2. There exists a set of mutually inequivalent knots $\{K_i\}_{i \in \mathbb{Z}}$ such that each $K_i$ satisfies the following conditions simultaneously:

(i) $\Delta_K(t) = 1$,
(ii) $\tau(K) = 1$,
(iii) $b(K) = 3$, and
(iv) $u(K) = 1$.

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Note that a tunnel number one knot is strongly invertible [1]. For any Laurent polynomial \( A(t) \) with \( A(1) = \pm 1 \) and \( A(t) \equiv A(t^{-1}) \), Kondo [8] and Sakai [17] constructed an unknotting number one knot whose Alexander polynomial is equal to \( A(t) \). And then Sakai [18] showed that his knots are strongly invertible.

Let \( F_g \) be a genus \( g \) surface standardly embedded in \( S^3 \). Any knot can be embedded in \( F_g \) for some \( g \). A torus knot is a knot which is embedded in \( F_1 \). According to the observation of C. Morin and M. Saito; cf. [13, p.138], a tunnel number one knot can be embedded in either \( F_1 \) or \( F_2 \).

From Theorem 1, we get the following result.

**Corollary.** For any Laurent polynomial \( A(t) \) with \( A(1) = \pm 1 \) and \( A(t) \equiv A(t^{-1}) \), there exists a knot embedded in \( F_2 \) whose Alexander polynomial is equal to \( A(t) \).

The Alexander polynomial of a torus knot has some restricted form. This is also true for the Alexander polynomial of a 2-bridge knot (cf. Hartley [5]). Sakuma proposed the following conjecture.

**Conjecture.** For any Laurent polynomial \( A(t) \) with \( A(1) = \pm 1 \) and \( A(t) \equiv A(t^{-1}) \), there exists a tunnel number one knot (resp. a 3-bridge knot) whose Alexander polynomial is equal to \( A(t) \).

Theorem 1 implies that this conjecture is true. The proofs are given in Sections 2 and 3; they are obtained by a small alteration of Sakai’s construction in [17].

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2. **Geometric indices**

To prove Theorems 1 and 2, we consider two classes of knots \( K(\alpha_1, \cdots, \alpha_n) \) and \( J(\alpha, \beta, \gamma) \) as illustrated in Figure 1.

**Figure 1**
Here $T_\alpha$ is a 2-string tangle with left hand $\alpha$-full twists, and $U_\alpha$ is a 4-string tangle with left hand $(2\alpha+1)$-half twists of two bands; see Figure 2. Note that “the band parts” of $U_\alpha$ are untwisted.

Then we easily see the upper bounds of the geometric indices.

**Lemma 1.** The unknotting numbers of $K(\alpha_1, \cdots, \alpha_n)$ and $J(\alpha, \beta, \gamma)$ are less than or equal to one.

**Proof.** Exchange the crossings $*$ in Figures 1a and 1b, and we get a trivial knot. □

**Lemma 2.** The tunnel numbers of $K(\alpha_1, \cdots, \alpha_n)$ and $J(\alpha, \beta, \gamma)$ are less than or equal to one.

**Proof.** Let $K = K(\alpha_1, \cdots, \alpha_n)$. For simplicity, we prove the lemma only in the case of $n = 4$. The proof in the general case is similar. We add the arc $\tau$ in $E(K)$ as illustrated in Figure 3a, and we have the deformations indicated in Figures 3a–3d. Then $\tau_2$ in Figure 3d is an unknotting tunnel of the 2-bridge knot; see [1]. Hence $\tau(K) \leq 1$.

For $J(\alpha, \beta, \gamma)$, the unknotting tunnel $\tau$ and the deformation are given in Figures 4a–4c. □

To state Lemma 3, we recall the following results.

**Proposition 1** ([12]). Let $K$ be a knot, and $\Sigma_2(K)$ the two-fold branched covering space of $S^3$ branched over $K$. If the unknotting number of $K$ is one, then there exists a strongly invertible knot $K'$ such that $\Sigma_2(K)$ can be obtained by the Dehn surgery of type $p/2$ on $K'$, where $p$ is an odd integer.

**Proposition 2** (Cyclic Surgery Theorem [3]). Let $K$ be a knot which is not a torus knot, and $K(r)$ the 3-manifold obtained by the Dehn surgery of type $r$ on $K$ ($r \in \mathbb{Q}$). If the fundamental group $\pi_1(K(r))$ is cyclic, then $r$ is an integer.

We denote by $C(\alpha_1, \cdots, \alpha_n)$ ($\alpha_i \in \mathbb{Z}$) the 2-bridge knot as shown in Figure 5; see [2].
Figure 3

Figure 4

Figure 5
Proposition 3 ([6]). We have the transformations of 2-bridge knots as follows:

\[ C(x_1, \ldots, x_n, a, y_1, \ldots, y_m) \]
\[ \cong C(x_1, \ldots, x_n + \epsilon, ||a||, (-1)^a(y + \epsilon), (-1)^ay_1, \ldots, (-1)^ay_m), \]
where \( \epsilon = a/||a|| \) and \( ||a|| = (-2\epsilon, 2\epsilon, \ldots, (-1)^{|a|-1}2\epsilon) \).

Lemma 3. Let \( K \) be one of \( K(\alpha_1, \ldots, \alpha_n) \) \((n \geq 3\) and \( \alpha_n \neq 0)\) and \( J(\alpha, \beta, \gamma) \) \((\beta \neq 0\ or \gamma \neq 0)\). Then \( b(K) = 3 \).

Proof. Case 1. Let \( K = K(\alpha_1, \ldots, \alpha_n) \) \((n \geq 3\) and \( \alpha_n \neq 0)\). It is easy to see that \( b(K) \leq 3 \). We show that \( b(K) \geq 3 \). If \( K \) is a 2-bridge knot, then the two-fold branched covering space of \( S^3 \) branched over \( K \), denoted by \( \Sigma_2(K) \), is a lens space. So \( \pi_1(\Sigma_2(K)) \) is cyclic.

From Proposition 1 and the construction given in [12], we have that \( \Sigma_2(K) \) is a 3-manifold obtained by Dehn surgery of type \( p/2 \) on the 2-bridge knot \( C = C(-1, -4a_2, -1, -4a_3, -1, \ldots, -4a_n) \), where \( p \) is an odd integer; see Figure 1b in Section 3. By Proposition 2, it is sufficient to prove that \( C \) is not a torus knot. If \( C \) is a torus knot, then \( C \cong \pm C(2, -2, \ldots, 2, -2) \) which is a torus knot of type \((2, m)\). Using Proposition 3, we deform \( C \) as follows:

\[ C(-1, -4a_2, -1, -4a_3, -1, -4a_4, -1, \ldots, -4a_n) \]
\[ \cong C((-1 + \epsilon_2) || -4a_2||, -1 + \epsilon_2, -4a_3, -1, -4a_4, -1, \ldots, -4a_n) \]
\[ \cong \begin{cases} 
C(0, -2, 2, \ldots, -2, 0, -4a_3, -1, -4a_4, -1, \ldots, -4a_n), & \text{if } \epsilon_2 = 1; \\
C(-2, -2, 2, \ldots, -2, -2, -4a_3, -1, -4a_4, -1, \ldots, -4a_n), & \text{if } \epsilon_2 = -1 \\
C(2, -2, \ldots, 2, -2 + 4a_3, 0, -2, 2, \ldots, -2, 0, \ldots, -4a_n), & \text{if } (\epsilon_2, \epsilon_4) = (1, 1); \\
C(-2, || -4a_2||, -2, 4a_3, 0, -2, 2, \ldots, -2, 0, \ldots, 4a_n), & \text{if } (\epsilon_2, \epsilon_4) = (1, -1); \\
C(-2, || -4a_2||, -2, 4a_3, -2, || -4a_4||, -2, \ldots, -4a_n), & \text{if } (\epsilon_2, \epsilon_4) = (-1, 1); \\
C(-2, || -4a_2||, -2, 4a_3, -2, || -4a_4||, -2, 2, \ldots, -4a_n), & \text{if } (\epsilon_2, \epsilon_4) = (-1, -1), \\
\end{cases} \]

where \( \epsilon_i = a_i/||a_i|| \) and \( ||a|| = (-2\epsilon, 2\epsilon, \ldots, (-1)^{|a|-1}2\epsilon) \). So \( C \) is not a torus knot.

Case 2. Let \( K = J(\alpha, \beta, \gamma) \), where \( \beta \neq 0 \) or \( \gamma \neq 0 \).

From Figure 1b, we see that \( b(K) \leq 3 \). Next we show that \( b(K) \geq 3 \). In the same way as in Case 1, we consider the two-fold branched covering space \( \Sigma_2(K) \)
of $S^3$ branched over $K$ which is obtained by Dehn surgery of type $p/2$ on the 2-bride knot $C = C(1, -4, -1, -4\beta, -1, -4\gamma)$, where $p$ is an odd integer. Using Proposition 3, we deform $C$ as follows:

\[
C(1, -4, -1, -4\beta, -1, -4\gamma) \\
\cong C(0, 2, -2, 2, -2, -4\beta, -1 + \epsilon_\gamma, || - 4\gamma||) \\
\cong C(-2, 2, -2, -4\beta, -1 + \epsilon_\gamma, || - 4\gamma||) \\
\neq \pm C(2, -2, \cdots, 2, -2),
\]

where $\epsilon_\gamma = -\gamma/|\gamma|$ and $|| - 4\gamma|| = (2\epsilon_\gamma, 2\epsilon_\gamma, \cdots, (-1)^{|4\gamma|-1}\epsilon_\gamma)$. From this, $C$ is not a torus knot. Hence $\pi_1(\Sigma_2(K))$ is non-cyclic, and thus $b(K) \geq 3$.

3. Alexander Polynomial

Now we calculate the Alexander polynomials of the above two classes of knots.

Case 1. $K(\alpha_1, \cdots, \alpha_n)$. We deform $K(\alpha_1, \cdots, \alpha_n)$ as illustrated in Figures 6a–6d. Thus $(S^3, K(\alpha_1, \cdots, \alpha_n))$ is homeomorphic to $(T(1), K)$ in Figures 6e, where $T(1)$ is obtained by performing the Dehn surgery of type $+1$ in $T$ (cf. [17]).

To calculate $\Delta_K(t)$, the Alexander polynomial of $K$, we use the infinite cyclic covering space $\tilde{E}(K)$ of the exterior of $K$, $E(K)$; see Chapter 7 in [16]. The Alexander module of $K$, $H_1(\tilde{E}(K); Z)$, is a $Z[t, t^{-1}]$-module, where $t$ acts in $\tilde{E}(K)$ as a covering translation. Let $\{\tilde{T}_i\}_{i \in \mathbb{Z}}$ be the system of the lifts of $T$ in $\tilde{E}(K)$, and $c_0$ the surgery coefficient of $\tilde{T}_i$. Then $\tilde{T}_j = t^j\tilde{T}_0$. We put $c_j = \ell k(\tilde{T}_0, \tilde{T}_j)$ $(j \neq 0)$. Then $c_j = c_{-j}$ for $j \geq 1$. Now $\tilde{E}(K)$ is constructed from $D^2 \times R^1$ by removing the regular neighborhoods of $\{\tilde{T}_i\}_{i \in \mathbb{Z}}$ and replacing them with solid tori, where the surgery coefficients are $c_0$. Then we have

\[
\Delta_K(t) = 1 - 2\alpha_1 + \alpha_1(t + t^{-1}) + \alpha_2(t^2 - 2t + 2 - 2t^{-1} + t^{-2}) \\
+ \sum_{i=3}^{n} \alpha_i(t^i - 2t^{i-1} + t^{i-2} + t^{i-2} - 2t^{-i+1} + t^{-i}).
\]

(4)

Let $(\alpha_1, \cdots, \alpha_n)$ be the solution of

\[
\begin{cases}
  c_n = \alpha_n, \\
  c_{n-1} = \alpha_{n-1} - 2\alpha_n, \\
  c_i = \alpha_i - 2\alpha_{i+1} + \alpha_{i+2} (1 \leq i \leq n-2), \\
  c_0 = 1 - 2\sum_{j=1}^{\infty} c_j.
\end{cases}
\]

Then, for any polynomial $A(t) = \sum_{j=-n}^{n} c_j t^j$ with $A(1) = 1$ and $c_j = c_{-j} (j \geq 1)$, the Alexander polynomial of $K(\alpha_1, \cdots, \alpha_n)$ is equal to $A(t)$.

As an example, we show it for the knot $K(-1, 1, -2)$ given in Figure 7. The infinite cyclic covering space is given in Figure 8.

Case 2. $J(\alpha, \beta, \gamma)$. We calculate the Alexander polynomial of $J(\alpha, \beta, \gamma)$ in the parallel way as in Case 1. $(S^3, J(\alpha, \beta, \gamma))$ is homeomorphic to $(T(1), K)$ in Figure 9c, where $T(1)$ is obtained by performing the Dehn surgery of type $+1$ in $T$. The infinite cyclic covering space of $\tilde{E}(J(\alpha, \beta, \gamma))$ is shown in Figure 10, where $\{\tilde{T}_i\}_{i \in \mathbb{Z}}$
Figure 8

Figure 9

Figure 10
is the system of the lifts of $T$. We see that

(i) $\text{lk}(\tilde{T}_0, \tilde{T}_1) = \alpha + 2\beta - 2\gamma - 1$,

(ii) $\text{lk}(\tilde{T}_0, \tilde{T}_2) = 1 + \gamma$, and

(iii) $\text{lk}(\tilde{T}_0, \tilde{T}_1) = 0$ for $i \geq 3$.

Therefore, we have

$$\Delta_{J(\alpha, \beta, \gamma)}(t) = 2\gamma - 2\alpha - 4\beta + 1 + (\alpha + 2\beta - 2\gamma - 1)(t + t^{-1}) + (\gamma + 1)(t^2 + t^{-2}).$$

If $A(t) = \sum_{i=-2}^{2} c_i t^i$, where $c_i = c_{-i}$ and $A(1) = 1$, then the Alexander polynomial of $J(\alpha, \beta, \gamma)$ is equal to $A(t)$, where $(\alpha, \beta, \gamma) = (c_1 + 2c_2 + 1, -1, c_2 - 1)$. In particular, $\Delta(J(-2\beta - 1, \beta, -1)) = 1$. We denote $J(-2\beta - 1, \beta, -1)$ by $K_\beta$.

To prove Lemma 4 below, we introduce a Laurent polynomial invariant ([7], Theorem 1.1), $c_0(K; x)$, determined by the following:

(i) $c_0(O; x) = 1, xc_0(L_+; x) - c_0(L_-; x) = c_0(L_0; x)$, where $O$ is a trivial knot, $L_+, L_-$ are knots and $L_0$ is a 2-component link, which are identical except near one point where they are as in Figure 11.

(ii) If $L = L_1 \cup L_2$ is a 2-component link with linking number $\lambda$, then

$$c_0(L; x) = (x - 1)x^{-\lambda}c_0(L_1; x)c_0(L_2; x).$$

This polynomial is a version of the first term of the skein polynomial [4], [11], [15].

**Lemma 4.** Each $K_\beta$ ($\beta \in \mathbb{Z}$) is non-trivial and $K_i \not\cong K_j$ if $i \neq j$.

**Proof:** From Figure 12, we calculate the Laurent polynomial invariant of $K_\beta$, $c_0(K_\beta; x)$. From (i), we have

$$xc_0(K_\beta; x) - 1 = (x - 1)x^0c_0(O; x)c_0(C(2, -2, -2, -2, 2, -2, 2); x),$$

and so, we have

$$c_0(K_\beta; x) = x^{-1} + (x - 1)x^{-1}c_0(C(2, -2, -2, -2, 2, 2); x) = x^{-3} - 3x^{-2} + 3x^{-1} + x^{3\beta - 2} - 3x^{3\beta - 1} + 3x^{\beta - 1} - x^{\beta + 1},$$

from which we obtain the result. \hfill \Box

**Concluding remark.** From Proposition 1.3 of [14], our two classes of knots have $(1, 1)$-decompositions. If a knot has a $(1, 1)$-decomposition, then its tunnel number is less than or equal to 1. The arc $\tau^*$ in Figure 3 is also an unknotted tunnel.

From the results of [9], [19], [20], an unknotted number 1 knot $K$ with $b(K) \leq 3$ is a hyperbolic knot or a trefoil knot. So our knots are hyperbolic knots.
REFERENCES


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