ON THE REALIZABILITY OF LEWY STRUCTURES

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Abstract. We prove that a nondegenerate CR structure with signature 
\((p, n-p)\) at \(0 \in \mathbb{R}^{2n+1}\) and with \(n\) first integrals \(z_1, \ldots, z_n\) satisfying 
\[d z_1 \wedge \bar{d} z_1 \wedge \cdots \wedge d z_n \wedge \bar{d} z_n \neq 0\]
is realizable if and only if an action of the group \(O(2p, 2n-2p)\) leaves invariant 
a one-dimensional subbundle of the structure bundle.

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The local realizability (embeddability) of CR structures has been the focus of several papers, among others we mention [Ak], [AH], [BR], [Ha], [HJ], [Ja1], [JT1, JT2, JT3, Ku, Ni]. It has been proved ([AH]) that abstract real analytic CR structures are realizable. It is also proved ([Ak], [Ku]) that \(C^\infty\) strictly pseudoconvex CR structures in \(\mathbb{R}^{2n+1}\) with \(n \geq 3\) are embeddable. On the other hand, there exist nonrealizable CR structures: of strictly pseudoconvex nature in \(\mathbb{R}^3\) ([Ni, JT1, JT2, JT3]); of Lewy type with signature \((1,n-1)\) in \(\mathbb{R}^{2n+1}\) ([JT1, JT2, JT3]); and of high codimension ([M1]).

In this note, we consider the realizability problem for a class of Lewy structures of hypersurface type at \(0 \in \mathbb{R}^{2n+1}\) with signature \((p, n-p)\). More precisely, we consider those structures with one first integral not listed. We prove that the “missing” first integral exists if and only if an action of the group \(O(2p, 2n-2p)\) leaves invariant a 1-dimensional subbundle of the Lewy bundle.

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A Lewy structure at \(0 \in \mathbb{R}^{2n+1}\) with signature \((p, n-p)\) is the datum of a subbundle \(C\) of \(CT^*U\), where \(U\) is a neighborhood of \(0 \in \mathbb{R}^{2n+1}\) and \(CT^*U\) is the complexified cotangent bundle \(T^*U\), such that \(C\) is generated by \(n+1\) smooth \(C^\infty\) one-forms \(\omega_1, \ldots, \omega_{n+1}\) satisfying
\[
\begin{align*}
\omega_1 \wedge \cdots \wedge \omega_n \wedge \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_n \wedge \omega_{n+1} & \neq 0, \\
\omega_1 \wedge \cdots \wedge \omega_{n+1} \wedge d\omega_k & = 0 \quad \text{for} \quad k = 1, \ldots, n+1,
\end{align*}
\]
and such that the Levi form has \(p\) positive and \(n-p\) negative eigenvalues. It follows from the formal integrability and the nondegeneracy of the Levi form that in suitable coordinates \((x_1, y_1, \ldots, x_n, y_n, u)\) of \(\mathbb{R}^{2n+1}\) the second jets of \(\omega_1, \ldots, \omega_{n+1}\)
are given by
\[ j^2 \omega_k = dz_k, \quad \text{for} \quad k = 1, \ldots, n, \]
(2)
\[ j^2 \omega_{n+1} = du + i \left( \sum_{l=1}^{p} z_l d\bar{z}_l - \sum_{l=p+1}^{n} z_l d\bar{z}_l \right), \]
where \( z_k = x_k + iy_k \). The Lewy structure is said to be realizable if the bundle \( \mathcal{C} \) is generated by exact forms. (For a comprehensive treatment of CR structures we refer to [Ja2] and [Tr].)

In this note, we are concerned with those Lewy structures \( \mathcal{L} \) with signature \((p, n-p)\) generated, in suitable coordinates, by the forms
\[ \omega_k = dz_k \quad \text{for} \quad k = 1, \ldots, n, \]
\[ \omega_{n+1} = du + \sum_{l=1}^{n} \lambda_l d\bar{z}_l. \]
(3)
Hence \( \omega_{n+1} \) satisfies
\[ d\omega_{n+1} \wedge dz_1 \wedge \cdots \wedge dz_n \wedge \omega_{n+1} = 0 \]
and its coefficients \( \lambda_l \) satisfy
\[ \lambda_l = i\epsilon_l z_l + o(|(z, u)|^2), \]
(4)
with \( \epsilon_l = 1 \) if \( l \leq p \) and \( \epsilon_l = -1 \) if \( l \geq p + 1 \). Notice that in order for the Lewy structure \( \mathcal{L} \) to be realizable, there must be \( C^\infty \) functions \( \mu_1, \ldots, \mu_n \) such that
\[ \omega_{n+1} + \mu_1 dz_1 + \cdots + \mu_n dz_n = gdf \]
for some \( C^\infty \) functions \( f \) and \( g \). The function \( f \) is called ([Ha], [HJ]) the missing first integral of \( \mathcal{L} \).

When the missing first integral \( f \) exists, the form \( df \) defines a Mizohata structure with signature \(|2n-4p|\) at \( 0 \in \mathbb{R}^{2n+1} \) (see [Tr] and [M2] for details about Mizohata structures). We know from [Tr] that there exists a germ of a diffeomorphism \( \Phi \) at \( 0 \in \mathbb{R}^{2n+1} \), tangent to the identity \( (D\Phi(0) = I) \), such that if \( \Phi = (\phi_1, \ldots, \phi_{2n+1}) \) and
\[ t_k = \phi_k(x, u) \quad \text{for} \quad k = 1, \ldots, 2n, \]
\[ s = \phi_{2n+1}(x, u), \]
then
\[ f \circ \Phi^{-1}(t, s) = s + i \left( \sum_{k=1}^{2p} t_k^2 - \sum_{k=2p+1}^{2n} t_k^2 \right). \]
(5)
Hence the invariance group \( O(2p, 2n-2p) \) of the quadratic form
\[ q(t) = \sum_{k=1}^{2p} t_k^2 - \sum_{k=2p+1}^{2n} t_k^2 \]
acting on \( \mathbb{R}^{2n+1} \) by
\[ A(t, s) = (At, s) \]
leaves invariant the bundle generated by $d(f \circ \Phi^{-1})$. Therefore the action of the group
\begin{equation}
G = \Phi^{-1}O(2p, 2n - 2p)\Phi
\end{equation}
leaves invariant the one-dimensional subbundle of $L$ generated by $df$.

Conversely, we prove that the existence of a group $G$, conjugate to $O(2p, 2n - 2p)$ under a diffeomorphism $\Phi$ with $D\Phi(0) = I$, and the existence of a one-dimensional subbundle $T$ of the Lewy bundle $L$ such that
\begin{equation}
G^* T = T
\end{equation}
is a sufficient condition for the realizability of the Lewy structure $L$.

Suppose that $T$ satisfying (8) is spanned by a form
\begin{equation}
\omega = a \omega_{n+1} + \sum_{k=1}^{n} b_k dz_k.
\end{equation}
We claim that $a(0) \neq 0$ and $b_k(0) = 0$ for $k = 1, \ldots, n$. Indeed, let $A \in O(2p, 2n - 2p)$ be defined by
\begin{align*}
A(t_1, \cdots, t_{2n}, s) &= (t_1, -t_2, \cdots, t_{2n-1}, -t_{2n}, s).
\end{align*}
Since $D\Phi(0) = I$, then $\Psi = \Phi^{-1} \circ A \circ \Phi$ satisfies
\begin{equation}
\Psi(x_1, y_1, \cdots, x_n, y_n, u) = (x_1, -y_1, \cdots, x_n, -y_n, u) + o(\|(z, u)\|).
\end{equation}
Thus,
\begin{equation}
\Psi^* \omega(0) = a(0) du + \sum_{k=1}^{n} b_k(0) d\bar{z}_k.
\end{equation}
This shows that for $\Psi^* \omega$ to be a section of $T$, it is necessary to have $a(0) \neq 0$ and $b_k(0) = 0$ as claimed.

After dividing $\omega$ by $a$, we see that $T$ is spanned by a form
\begin{equation}
\Omega = \omega_{n+1} + \sum_{k=1}^{n} \mu_k dz_k
\end{equation}
\begin{equation}
= du + \sum_{k=1}^{n} \lambda_k d\bar{z}_k + \sum_{k=1}^{n} \mu_k dz_k.
\end{equation}
Now, observe that since
\begin{equation}
\Psi^* \Omega \wedge \Omega = 0,
\end{equation}
($\Psi$ is defined in (10)) and since the $\lambda_k$’s satisfy (4), it follows that necessarily
\begin{equation}
\mu_k = i \epsilon_k \bar{z}_k + o(\|(z, u)\|^2).
\end{equation}
This means that the first jet of $\Omega$ is
\begin{equation}
\mathcal{J}^1 \Omega = du + i \sum_{k=1}^{n} \epsilon_k d(z_k \bar{z}_k).
\end{equation}
With this, our aim would be acheived if we could show that
\begin{equation}
\Omega \wedge d\Omega = 0.
\end{equation}
To prove the formal integrability (16), we use again the change of coordinates $\Phi$ and write

$$ (\Phi^{-1})^* \Omega = \alpha(t, s) \left[ ds + \sum_{k=1}^{2n} a_k(t, s) dt_k \right] = \alpha(t, s) \Theta. $$

Since the bundle generated by the form $\Theta$ is invariant under the action of the group $O(2p, 2n - 2p)$, then it is not difficult to see that

$$ A^* \Theta = \Theta \quad \forall A \in O(2p, 2n - 2p). $$

To prove that $\Theta$ is formally integrable, it is enough to prove that it descends to the orbit space of two variables. For this, we consider the hyperbolic radius

$$(25) \quad r = \rho \cosh t - r_2 \sinh t,$$

in the coordinates $(\rho, t)$ the form $\Gamma = U_1 dr_1 + U_2 dr_2$ has the expression

$$(25) \quad \Gamma = (U_1 \cosh t + U_2 \sinh t) dp + (U_1 \sinh t + U_2 \cosh t) dt$$

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From (19), it follows that $\Gamma$ is invariant under the translations $t \mapsto t + \tau$ and under the reflection $t \mapsto -t$. The invariance under translations implies that $V$ and $W$ are independent of $t$, and then the invariance under the reflection gives $W = 0$. This proves (23).

To complete the proof of the lemma, we go back to (21) and prove that $\Lambda = 0$. Again from (19) it follows that $A^* \Lambda = \Lambda$ for every $A \in O(2p, 2n - 2p)$. Let $\Lambda = \Lambda_1 + \Lambda_2$, where $\Lambda_1$ is a 1-form on $S^{2p-1}$ with parameters $s, r_1, r_2, \theta^2$ and $\Lambda_2$ is a form on $S^{2n-2p-1}$ with parameters $s, r_1, r_2, \theta^1$. We deduce that

$$A_1^* \Lambda_1 = \Lambda_1 \quad \text{and} \quad A_2^* \Lambda_2 = \Lambda_2$$

for every $A_1 \in O(2p)$ and every $A_2 \in O(2n - 2p)$.

Now suppose that $\alpha = a(x, y)dx + b(x, y)dy$ is a 1-form in a two-dimensional plane satisfying $B^* \alpha = \alpha$ for every $B \in O(2)$. We write $\alpha$ in polar coordinates $(r, \phi)$ as

$$\alpha = (a \cos \phi + b \sin \phi)dr + r(-a \sin \phi + b \cos \phi)d\phi$$

$$= p(r, \phi)dr + q(r, \phi)d\phi.$$ 

Since $\alpha$ is invariant under the translations $\phi \mapsto \phi + \tau$, it follows at once that $p$ and $q$ are independent of $\phi$. The invariance of $\alpha$ under the reflection $\phi \mapsto -\phi$ shows that $q = 0$. That is,

$$\alpha = p(r)dr.$$ 

With this at hand, we prove that $\Lambda_1 = 0$ as follows. Take any point $p_0 \in S^{2p-1}$ and consider the stereographic projection with center $p_0$. Consider the action of the group $O(2)$ on the chart $\mathbb{R}^{2p-1} = \mathbb{R}^2 \times \mathbb{R}^{2p-3}$, with $p_0$ as the origin, as $(x, x') \mapsto (Bx, x')$. The form $\Lambda_1$ is invariant under this action and so it follows from (28) that $\Lambda_1$ is independent of the polar angle $\phi$ in $\mathbb{R}^2$. Since $p_0$ is arbitrary in $S^{2p-1}$, the nullity of $\Lambda_1$ follows. Similarly we prove that $\Lambda_2 = 0$. The lemma is proved.

In conclusion, we have proved the following

**Theorem.** A Lewy structure $\mathcal{L}$ with signature $(p, n-p)$ and with only one “missing” first integral is realizable if and only if there exist a germ of a diffeomorphism $\Phi$ at $0 \in \mathbb{R}^{2n+1}$, tangent to the identity, and a 1-dimensional subbundle $\mathcal{T}$ of the Lewy bundle such that

$$\Psi^* \mathcal{T} = \mathcal{T} \quad \forall \Psi \in \Phi^{-1}O(2p, 2n - 2p)\Phi.$$

**References**


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