ON C*-ALGEBRAS ASSOCIATED WITH LOCALLY COMPACT GROUPS

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Abstract. Let $G$ be a locally compact group, and let $G_d$ denote the same group $G$ with the discrete topology. There are various C*-algebras associated to $G$ and $G_d$. We are concerned with the question of when these C*-algebras are isomorphic. This is intimately related to amenability. The results can be reformulated in terms of Fourier and Fourier-Stieltjes algebras and of weak containment properties of unitary representations.

Introduction

Let $G$ be a locally compact group and $G_d$ the group $G$ equipped with the discrete topology. Denote by $\lambda_G$ and $\lambda_{G_d}$ the left regular representation of $G$ and $G_d$ on $L^2(G)$ and $\ell^2(G_d)$, respectively. Let $C^*_\delta(G)$ and $C^*_r(G_d)$ be the C*-subalgebras of $\mathcal{L}(L^2(G))$ and $\mathcal{L}(\ell^2(G_d))$ generated by $\lambda_G(G)$ and $\lambda_{G_d}(G_d)$, respectively.

When $G$ is abelian, $C^*_r(G_d)$ via Fourier transform is isomorphic to the algebra $C(\hat{G}_d)$ of all continuous functions on the compact dual group $\hat{G}_d$ of $G_d$. Similarly, $C^*_\delta(G)$ is isomorphic to the norm closed subalgebra of $L^\infty(\mathcal{G})$ generated by the elements of $G_d$ considered as functions on $\Gamma = \hat{G}$ by means of the canonical isomorphism between $G$ and $\hat{\hat{G}}$. Moreover, the algebra $C(\hat{\mathcal{G}})$ of almost periodic functions on $\Gamma$ considered as a subalgebra of $L^\infty(\hat{\mathcal{G}})$ is generated by the set $G_d = \hat{\hat{G}}_d = \hat{\hat{b}}\Gamma$ of continuous characters on $\Gamma$. Hence, $C^*_\delta(G)$ is isomorphic to $C^*_r(G_d)$.

In this paper, we shall be concerned with possible extensions of this result to non-abelian groups.

Relationships between the two unital C*-algebras $C^*_\delta(G)$ and $C^*_r(G_d)$ were studied by Dunkl and Ramirez [DuR] and by Bédos [Béd]. It was shown in [DuR], Theorem 2.5 (see also [Béd], Lemma 2) that $\lambda_{G_d}$ extends to a (surjective) *-homomorphism

$$\Phi : C^*_\delta(G) \rightarrow C^*_r(G_d).$$

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This means that $\lambda_G$, when viewed as a unitary representation of $G_d$, weakly contains $\lambda_{G_d}$ (concerning the notion of weak containment, we refer to [Dix], Chap. 18). For the convenience of the reader, we reproduce below (Proposition 1) a very short proof of this fact appearing in [BeV], Proposition 1.

A natural question is whether $\Phi$ is an isomorphism (that is, whether $\Phi$ is injective). This is easily settled in case $G$ is amenable (see also [DuR], [Béd]): $\Phi$ is an isomorphism if and only if $G_d$ is amenable. Indeed, in this case, by Hulanicki’s theorem (see [Pat], Theorem 4.21 or [Pie], Theorem 8.9) the trivial one-dimensional representation $1_G$ is weakly contained in $\lambda_G$ (as representations of $G$ and, a fortiori, as representations of $G_d$). The claim now follows, as $G_d$ is amenable if and only if $1_G$ is weakly contained in $\lambda_{G_d}$.

Our main result in this paper is a complete answer to the question raised above.

**Theorem 1.** Let $G$ be a locally compact group. Then $\Phi : C^*_v(G) \rightarrow C^*_v(G_d)$ is an isomorphism if and only if $G$ contains an open subgroup $H$ such that $H_d$ is amenable.

It is well-known that, for a connected Lie group $G$, $G_d$ is amenable if and only if $G$ is solvable. So, the following is an immediate consequence of Theorem 1.

**Corollary 1.** Let $G$ be a connected Lie group. Then $\Phi$ is an isomorphism if and only if $G$ is solvable.

In terms of weak containment, Theorem 1 states that $\lambda_G$ is weakly contained in $\lambda_{G_d}$ if and only if $G$ contains an open subgroup $H$ such that $H_d$ is amenable. In fact, we shall prove the following stronger result:

**Theorem 2.** Let $G$ be a locally compact group. The following are equivalent:

(i) $\lambda_{G_d}$ weakly contains $\lambda_G$;

(ii) $\lambda_{G_d}$ weakly contains some continuous unitary representation of $G$, viewed as representation of $G_d$;

(iii) $G$ contains an open subgroup $H$ such that $H_d$ is amenable.

As is to be expected, the difficult part in the proof of this theorem is to show that (ii) implies (iii). This will require several steps, the main one being the case where $G$ is a connected Lie group. The proof in this situation is based on the existence, for $G$ non-solvable, of many non-abelian free subgroups of $G$ and on estimates for the norm of convolution operators on free groups, similar to those appearing in [Kes], [Lei] and [AkO].

We now reformulate our results in terms of the Fourier and Fourier-Stieltjes algebras of $G$ and $G_d$. Recall that the Fourier-Stieltjes algebra $B(G)$ of a locally compact group $G$ is the linear span of all continuous positive definite functions on $G$. Recall also that $B(G)$ may be identified, in a natural way, with the dual of $C^*(G)$, the $C^*$-algebra of $G$. The Fourier algebra $A(G)$ of $G$ is the closed subalgebra generated by all functions in $B(G)$ with compact support. $A(G)$ may also be described as the set of all matrix coefficients of the regular representation $\lambda_G$ of $G$. More details on $A(G)$ and $B(G)$ are to be found in [Eym] where these spaces are extensively studied.

Let $\overline{A(G_d)}^{\ast\ast}$ denote the closure of $A(G_d)$ in $B(G_d)$, with respect to the weak$^*$ topology $\sigma(B(G_d), C^*(G_d))$.

Then $\overline{A(G_d)}^{\ast\ast}$ coincides with the space, denoted by $B_{\lambda_d}(G_d)$ in [Eym], of all matrix coefficients of the unitary representations of $G_d$ which are the linear span
of the positive definite functions associated with the unitary representations which are weakly contained in $\lambda_{G_d}$ (see [Eym], (2.1) Proposition). Hence, the equality $\overline{A(G_d)}^\ast = B_{\lambda_d}(G_d)$ follows from [Eym], (1.2.1) Proposition.

As any continuous function in $B(G_d)$ actually lies in $B(G)$ ([Eym], (2.24) Corollaire 1), it is now clear that Theorem 2 may be reformulated as follows.

**Theorem 2’.** Let $G$ be a locally compact group. The following are equivalent:

(i) $\overline{A(G_d)}^\ast$ contains $A(G)$;

(ii) $\overline{A(G_d)}^\ast$ contains a non-zero continuous function on $G$;

(iii) $G$ contains an open subgroup $H$ such that $H_d$ is amenable.

This paper is organized as follows. In Section 1, we treat the case of connected Lie groups. The proof of Theorem 2 is then completed in Section 2. Section 3 contains some remarks about other $C^\ast$-algebras associated with $G$.

1. THE CONNECTED LIE GROUP CASE

In this section, we show that (ii) implies (iii) in Theorem 2 when $G$ is a connected Lie group.

As mentioned above, we first reproduce the proof given in [BeV] for the existence of $\Phi$.

**Proposition 1.** $\lambda_{G_d}$ is weakly contained in $\lambda_G$, for any locally compact group $G$.

**Proof.** Since $\lambda_{G_d}$ is cyclic, it suffices to show that $\delta_e$, the Dirac function at the group unit $e$, is the pointwise limit of positive definite functions associated to $\lambda_G$. Let $F$ be a finite subset of $G \setminus \{e\}$. Choose a neighbourhood $K$ of $e$ such that $gK \cap K = \emptyset$ for all $g \in F$. Set

$$\varphi(g) = \frac{1}{\mu(K)} \langle \lambda_G(g)\chi_K, \chi_K \rangle, \quad g \in G,$$

where $\chi_K$ is the characteristic function of $K$, and $\mu$ is a left Haar measure on $G$. Then $\varphi$ is a positive definite function associated to $\lambda_G$ such that $\varphi(g) = \delta_e(g)$ for all $g \in F \cup \{e\}$. \hfill $\square$

We shall use the following estimate for the norm of convolution operators on free groups. This may easily be deduced from more general results appearing in [AkO], [Kes] and [Lei]. For the sake of completeness, we prefer to give an independent, short proof (compare also [BCH], Proof of Lemma 2.2).

**Proposition 2.** Let $\Gamma$ be a non-abelian free group on the generators $x$ and $y$, and let $\lambda$ denote the left regular representation of $\Gamma$ on $\ell^2(\Gamma)$. Then

$$\| \sum_{n=1}^\infty a_n \lambda(y^n x y^{-n}) \| \leq 2\|a\|_2$$

for all sequences $a = (a_n)_{n \in \mathbb{N}}$ in $\ell^2(\mathbb{N})$.

**Proof.** Let $W_0$ be the subset of $\Gamma$ consisting of the words which do not begin with a nontrivial power of $y$. Let $W_n = y^n W_0$ for $n \in \mathbb{N}$. Then $W_n \cap W_m = \emptyset$ for all $n \neq m$. For any $f, g \in \ell^2(\Gamma)$ and $n \in \mathbb{N}$, the following holds

$$|\langle \lambda(y^n x y^{-n})f, g \rangle| = |\langle \lambda(y^n x y^{-n})\chi_{W_n} f, g \rangle| + |\langle \lambda(y^n x y^{-n})\chi_{\Gamma \setminus W_n} f, g \rangle|$$

$$\leq \|\chi_{W_n} f\| \|g\| + \|f\| \|\chi_{y^n x y^{-n} \Gamma \setminus W_n} g\|.$$
where $\chi_A$ denotes the characteristic function of $A \subseteq \Gamma$. Since $y^nxy^{-n}(\Gamma \setminus W_n) \subseteq W_n$, we get

$$|\langle \lambda(y^nxy^{-n})f, g \rangle| \leq \|\chi_{W_n}f\|\|g\| + \|f\|\|\chi_{W_n}g\|$$

and therefore

$$\left|\sum_{n=1}^{\infty} a_n \lambda(y^nxy^{-n})f, g \right| \leq \sum_{n=1}^{\infty} |a_n|\left(\|\chi_{W_n}f\|\|g\| + \|f\|\|\chi_{W_n}g\|\right)$$

$$\leq 2\|a\|_2\|f\|\|g\|,$$

by the Cauchy-Schwarz inequality.

**Proposition 3.** Let $G$ be a connected non-solvable Lie group. Then no continuous unitary representation of $G$, viewed as a representation of $G_d$, is weakly contained in $\lambda_{G_d}$.

**Proof.** It is well-known that such a group contains a non-abelian free subgroup $F$ on two generators $a$ and $b$ (see [Pat], Theorem 3.9). For any finite set of integers $i_1, \ldots, i_n \in \mathbb{Z}\setminus\{0\}$, let $p_{i_1,\ldots,i_n}: G \to G$ denote the word function

$$p_{i_1,\ldots,i_n}(x) = \begin{cases} a_1x_i^2\ldots a_1^{n-1}x_i^{i_n}, & \text{if } n \text{ is even,} \\ a_1x_i^2\ldots x_i^{n-1}a_i^{i_n}, & \text{if } n \text{ is odd.} \end{cases}$$

Then, the set

$$G_{i_1,\ldots,i_n} = \{x \in G, p_{i_1,\ldots,i_n}(x) \neq e\}$$

is open. It is also nonempty since $b \in G_{i_1,\ldots,i_n}$. Moreover, because $p_{i_1,\ldots,i_n}$ is an analytic function on $G$, $G_{i_1,\ldots,i_n}$ is dense. So, by Baire’s category theorem, the intersection

$$X = \bigcap\{G_{i_1,\ldots,i_n}; i_1, \ldots, i_n \in \mathbb{Z}\setminus\{0\}\}$$

is dense in $G$.

By the definition of $X$, for any $x \in X$, the subgroup $\Gamma_x$ generated by $a$ and $x$ is a free group. Hence, by Proposition 2, for any $x \in X$ and $N \in \mathbb{N}$, we have the following estimate:

$$(*) \quad \left\| \frac{1}{N} \sum_{n=1}^{N} \lambda_{\Gamma_x}(a^nxa^{-n}) \right\| \leq \frac{2}{\sqrt{N}}$$

where $\lambda_{\Gamma_x}$ denotes the regular representation of the discrete group $\Gamma_x$. But, since the restriction of $\lambda_{G_d}$ to $\Gamma_x$ is a multiple of $\lambda_{\Gamma_x}$, we may replace $\lambda_{\Gamma_x}$ by $\lambda_{G_d}$ in the above inequality $(*)$.

Now suppose, by contradiction, that there exists a continuous unitary representation $\pi$ of $G$ which is weakly contained in $\lambda_{G_d}$. Then, by $(*)$,

$$\left\| \frac{1}{N} \sum_{n=1}^{N} \pi(a^nxa^{-n}) \right\| \leq \frac{2}{\sqrt{N}}$$
for any $x \in X$ and any $N \in \mathbb{N}$. Therefore, for any unit vector $\xi$ in the Hilbert space of $\pi$, we have

$$(**): \quad \left| \frac{1}{N} \sum_{n=1}^{N} (\pi(a^n x a^{-n}) \xi, \xi) \right| \leq \frac{2}{\sqrt{N}}$$

for all $x \in X$ and $N \in \mathbb{N}$. Since $X$ is dense and $\pi$ is (strongly) continuous, $(**)$ holds for any $x \in G$.

Taking $x = e$ and $N \geq 5$, we reach the contradiction

$$1 = \frac{1}{N} \sum_{n=1}^{N} (\pi(a^n e a^{-n}) \xi, \xi) \leq \frac{2}{\sqrt{5}}.$$  

2. The general case

We now proceed with the proof of Theorem 2. This will require two results related to Proposition 3.

**Proposition 4.** Let $G$ be an amenable locally compact group. Assume that $\lambda_{G_d}$ weakly contains some continuous unitary representation $\pi$ of $G$. Then $G_d$ is amenable.

**Proof.** Let $\bar{\pi}$ denote the representation conjugate to $\pi$. Then the (inner) tensor product $\pi \otimes \bar{\pi}$ is weakly contained in $\lambda_{G_d} \otimes \lambda_{G_d}$ and, hence, in $\lambda_{G_d}$, since $\lambda_{G_d} \otimes \lambda_{G_d}$ is a multiple of $\lambda_{G_d}$.

On the other hand, because $G$ is amenable, the trivial representation $1_G$ is weakly contained in $\pi \otimes \bar{\pi}$, by [Bek], Theorems 2.2 and 5.1. Hence, $1_G$ is weakly contained in $\lambda_{G_d}$. Therefore, $G_d$ is amenable.

Next, we extend Proposition 3 to all connected groups.

**Proposition 5.** Let $G$ be a connected locally compact group. Assume that $\lambda_{G_d}$ weakly contains some continuous unitary representation $\pi$ of $G$. Then $G_d$ is amenable.

**Proof.** By the structure theory for connected groups, $G$ contains a compact normal subgroup $K$ such that $G/K$ is a Lie group (see [MoZ], p. 175).

We first claim that $K_d$ is amenable. Indeed, since the restriction $\lambda_{G_d}|_{K}$ of $\lambda_{G_d}$ to $K$ is a multiple of $\lambda_{K_d}$, $\lambda_{K_d}$ weakly contains $\pi|_{K}$. As $K$ is amenable, the claim follows from Proposition 4.

Suppose, by contradiction, that $(G/K)_d$ is not amenable. Then, by the proof of Proposition 3, there exist an element $\dot{a} \in G/K$ and a dense subset $\dot{X}$ of $G/K$ such that, for any $\dot{x} \in \dot{X}$, the subgroup generated by $\dot{a}$ and $\dot{x}$ is free.

Let $p : G \to G/K$ denote the canonical projection. Choose any $a \in G$ with $p(a) = \dot{a}$, and set $X = p^{-1}(\dot{X})$. Then, for any $x \in X$, the subgroup of $G$ generated by $a$ and $x$ is free. As in the proof of Proposition 3, this, together with the fact that $X$ is dense in $G$, yields a contradiction.

Therefore, $(G/K)_d$ and $K_d$ are amenable. It follows that $G_d$ is amenable.

**Proof of Theorem 2.** That (i) implies (ii) is obvious. Suppose (ii) holds, that is, $\lambda_{G_d}$ weakly contains some continuous representation $\pi$ of $G$. Let $G^0$ denote the connected component of $e$ in $G$. As $\lambda_{G_d}$ weakly contains $\pi|_{G^0}$, $G^0_d$ is amenable by
Proposition 5. Since \( G/G_0 \) is totally disconnected, we may choose an open subgroup \( H \) of \( G \) containing \( G_0 \) such that \( H/G_0 \) is compact (see [HeR], Theorem 7.7). We claim that \( H \) is amenable. Indeed since \( H \) is amenable, the claim follows immediately from Proposition 4. This completes the proof that (ii) implies (iii).

Suppose (iii) holds, that is, \( G \) contains an open subgroup \( H \) such that \( H \) is amenable. Then \( \lambda_H | H_d \) weakly contains \( \lambda_H \) (in fact, \( \lambda_H | H_d \) weakly contains any unitary representation of \( H_d \)).

Now, since \( G/H \) is discrete, the induced representation \( \text{ind}^{G_d}_{H_d} \lambda_H \) is equivalent to \( \text{ind}^{G_d}_H \lambda_H = \lambda_G \). Therefore, by continuity of inducing, \( \lambda_G \) is weakly contained in \( \text{ind}^{G_d}_H \lambda_H = \lambda_{G_d} \). This shows that (i) holds and completes the proof of Theorem 2.

3. Some remarks on other \( C^* \)-algebras associated with \( G \)

Let \( \omega \) be the universal representation of the locally compact group \( G \). Let \( C^*_\omega(G) \) be the (maximal) \( C^* \)-algebra of \( G \), and let \( C^*_\delta,\omega(G) \) denote the \( C^* \)-algebra generated by all operators \( \omega(x) \), \( x \in G \). There is an obvious surjective \(*\)-homomorphism
\[ \Psi : C^*_\delta,\omega(G) \to \overset{\circ}{C^*_\delta,\omega}(G). \]

The main result in [BeV] may be reformulated as follows.

**Theorem 3 ([BeV]).** Let \( G \) be a connected Lie group. Then \( \Psi \) is an isomorphism if and only if \( G \) is solvable.

**Remark 1.** We do not know how to extend this result to other locally compact groups. A reasonable conjecture seems to be: \( \Psi \) is an isomorphism if and only if \( G \) contains an open subgroup \( H \) such that \( H \) is amenable (that is, by Theorem 1, if and only if \( \Phi \) is an isomorphism).

Let \( \Lambda : C^*_\delta,\omega(G) \to C^*_\delta(G) \) be the surjective \(*\)-homomorphism, defined in the natural way. The following result, for which we offer a simple proof, may also be deduced from [Béd, Theorem 1].

**Theorem 4.** Let \( G \) be a locally compact group. Then \( \Lambda \) is an isomorphism if and only if \( G \) is amenable.

**Proof.** If \( G \) is amenable, then \( \lambda_G \) is weakly equivalent to the universal representation \( \omega \) (even as representations of \( G \)). This implies that \( \Lambda \) is an isomorphism. Conversely, assume \( \Lambda \) is injective. Then \( 1_G \) is weakly contained in \( \lambda_G \), where both representations are viewed as representations of \( G_d \). That is, \( G \) has Reiter’s weak property \((P_2^*)\) (see [Pie], p. 56) which is known to characterise the amenability of \( G \).

**Remark 2.** We used in the above proof the surprising fact that Reiter’s weak property \((P_2^*)\) is equivalent to the stronger property \((P_2)\) (saying that \( 1_G \) is weakly contained in \( \lambda_G \), as representations of \( G \)) and, hence, to the amenability of \( G \). Usually, the proof of this equivalence is a long and tedious one involving topological invariant means (compare [Pie], p. 56). It is worth mentioning that the argument used by Bédos in [Béd], Proof of Theorem 1, provides a quick and elegant proof for this equivalence. Indeed, assume \( 1_G \) is weakly contained in \( \lambda_G \), as representations...
of $G_d$. Then $1_G$ defines a state $\varphi$ on $C^*_\delta(G)$ such that $\varphi(\lambda_G(x)) = 1$ for all $x \in G$. Extend $\varphi$ to a state $\bar{\varphi}$ on $L(L^2(G))$. Cauchy-Schwarz inequality shows that

$$\bar{\varphi}(\lambda_G(x)T) = \bar{\varphi}(T\lambda_G(x))$$

for all $x \in G, T \in L(L^2(G))$.

Then, denoting by $M_f$ the multiplication operator on $L^2(G)$ by $f \in L^\infty(G)$, one defines a mean $\mu$ on $L^\infty(G)$ as follows:

$$\mu(f) = \bar{\varphi}(M_f), \quad \forall f \in L^\infty(G).$$

Since

$$M_{\lambda_G(x)f} = \lambda_G(x)M_f \lambda_G(x^{-1}), \quad \forall x \in G, f \in L^\infty(G),$$

$\mu$ is left invariant. This shows that $G$ is amenable.

It should be observed that, using the notion of amenable representations as defined in [Bek], the above argument shows that $\lambda_G$ is amenable.

The following corollary is also proved in [DuR], Proposition 3.2.

**Corollary 2.** Let $G$ be a locally compact group. Then

$$\Phi \circ \Lambda: C^*_{\delta,\omega}(G) \to C^*_r(G_d)$$

is an isomorphism if and only if $G_d$ is amenable.

**Proof.** If $G_d$ is amenable, then, as shown earlier, $\Lambda$ and $\Phi$ are isomorphisms.

Conversely, suppose $\Phi \circ \Lambda$ is an isomorphism. Since $\Lambda$ is an isomorphism, by Theorem 4, $G$ is amenable. As mentioned in the introduction, this implies that $G_d$ is amenable, as $\Phi$ is an isomorphism.

\[ \square \]

**References**


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