CONTINUOUS SINGULAR MEASURES
WITH ABSOLUTELY CONTINUOUS CONVOLUTION SQUARES

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ABSTRACT. We prove for every non-abelian compact connected group \( G \) there is a continuous, singular, central measure \( \mu \) with \( \mu * \mu \in L^p \) for all \( p, 1 \leq p < \infty \). We also construct such measures on some families of non-abelian compact totally disconnected groups. These results settle an open question of Ragozin.

1. Introduction

In 1966, Hewitt and Zuckerman [6] proved that if \( G \) is a non-discrete locally compact abelian group with Haar measure \( \lambda \), then there exists a non-negative, continuous, regular measure \( \mu \) on \( G \) that is singular to \( \lambda(\mu \perp \lambda) \) such that \( \mu(G) = 1 \), \( \mu * \mu \) is absolutely continuous with respect to \( \lambda(\mu * \mu \ll \lambda) \) and the Lebesgue-Radon-Nikodym derivative of \( \mu * \mu \) with respect to \( \lambda \) is in \( L^p(G,\lambda) \) for all \( p, 1 \leq p < \infty \). This result was partially generalised to non-abelian connected compact simple groups by Ragozin [10] and to any locally compact group (nondiscrete) by Karanikas and Koumandos [7]. They showed that there exist continuous singular measures \( \mu \) with \( \mu * \mu \in L^1(G,\lambda) \), \( \lambda \) being the left Haar measure on \( G \). Ragozin [10] mentioned that even for connected compact simple Lie groups it is an open question whether there exists a singular measure \( \mu \) with \( \mu * \mu \in L^p(G,\lambda) \) for all \( p, 1 \leq p < \infty \).

In this paper, we show that on any non-abelian compact, connected group or on any compact Lie group \( G \), there exist central, continuous, singular measures \( \mu \) with \( \mu * \mu \in L^p(G,\lambda) \) for all \( p, 1 \leq p < \infty \) and also prove this result on some specific disconnected non-abelian compact groups. We also note two remarkable but difficult works of Saeki [11], [12] and a paper of Stempak [14] on this subject.

In section 2, we establish our notation. We state and prove our main theorem (Theorem 3.1) in section 3. Section 4 contains the results on some specific totally disconnected compact groups.

2. Notation

\( G \) always denotes a non-abelian compact group unless stated otherwise. Let \( G \) be a compact connected Lie group and \( T = T^\ell \) be a maximal torus for \( G \). Let \( R \) denote the set of roots of \( G \) with respect to \( T \), \( \Delta \) a base for \( R \), and \( R_+ \) the set of positive roots with respect to \( \Delta \).
The Weyl group $W$ is the finite group $N_G(T)/T$, where $N_G(T)$ is the normaliser of $T$ in $G$ and one has an action on $T$ given by
\[ w = gT : t \mapsto wt = gtg^{-1}. \]
For each $w \in W$, $H \to dw(e)H$ defines an action of $w$ on the Lie algebra $\mathfrak{t}$ of $T$ which we denote also by $H \to wH$. Let $\text{sgn}(w)$ denote the determinant of $dw(e)$.

Let $\mathfrak{g}^*$ and $\mathfrak{t}^*$ denote the duals of the real Lie algebras $\mathfrak{g}$ and $\mathfrak{t}$ respectively. The derivatives of the roots of $G$ on $T$ are purely imaginary linear forms on $\mathfrak{t}$. For $\alpha \in R$, let $d\alpha(e) = i\hat{\alpha}$ where $\hat{\alpha} \in \mathfrak{t}^*$. Let $(,)$ denote the Killing form on $\mathfrak{t}$, and by duality transfer it to a form on $\mathfrak{t}^*$. For $H_1, H_2 \in \mathfrak{t}$, we denote $\frac{2\langle H_1, H_2 \rangle}{\langle H_2, H_2 \rangle}$ by $\langle H_1, H_2 \rangle$.

For $p \in \mathfrak{t}^*$, we denote by $H_p \in \mathfrak{t}$ the element satisfying
\[ p(H) = (H_p, H) \quad \forall H \in \mathfrak{t}. \]
Let $\triangle = \{\alpha_1, \ldots, \alpha_\ell\}$. We denote by $\tilde{\alpha}_i = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}, 1 \leq i \leq \ell$.

Let $\delta = 1/2 \sum_{\alpha \in R_+} \tilde{\alpha}$ and $q = \prod_{\alpha \in R_+} (1 - \tilde{\alpha})$. Define $j$, an Ad-invariant function on $g$ as follows: for $H \in \mathfrak{t}$, set
\[ j(H) = e^{i\delta(H)}q(\exp H) \prod_{\alpha \in R_+} i\tilde{\alpha}(H). \]
Then $|j|^2$ is exactly the determinant of the exponential map, and in fact $j$ takes real values.

Now we define the wrapping map $\Phi$ [2]. Let $\mu \in M(g)$ be an Ad-invariant measure of compact support. Define
\[ \langle \Phi \mu, \varphi \rangle = \langle \mu, j\tilde{\varphi} \rangle \quad \forall \varphi \in C(G) \]
where
\[ \tilde{\varphi}(X) = \varphi(\exp X). \]
It follows easily that $\Phi \mu$ is a central measure on $G$ and if $\mu$ is continuous then $\Phi \mu$ is continuous too.

Also for $\sigma \in \hat{G}$, $\langle \Phi \mu, \hat{\sigma} \rangle = \tilde{\mu}(\lambda + \delta)I_{d\sigma}$, where $e^{i\lambda}$, $\lambda \in \mathfrak{t}^*$ is the highest weight of $\sigma$ [2].

3. THE MAIN THEOREM

In this section we state the main result (Theorem 3.1) of this paper and prove it.

**Theorem 3.1.** Let $G$ be a compact connected group or a compact Lie group. Then there exists a central, singular, continuous measure $\mu \in M(G)$ such that $\mu * \mu$ is absolutely continuous and the Lebesgue-Radon-Nikodym derivative of $\mu * \mu$ is in $L^p(G)$ for all $p$, $1 \leq p < \infty$.

To prove Theorem 3.1, we will show that for $G$ satisfying the hypothesis of Theorem 3.1, there exists a central, singular, continuous measure $\mu$ defined on $G$ whose Fourier-Stieltjes transform $\hat{\mu}$ is $p$-summable for every $p > 2$. Then $\mu * \mu$ is absolutely continuous and the Lebesgue-Radon-Nikodym derivative of $\mu * \mu$ is in $L^p$ for all $p$, $1 \leq p < \infty$ by the Hausdorff-Young theorem ([4], Theorem 31.24).
Proof of Theorem 3.1 for $G$ compact simply connected semisimple Lie group. Let 
\{ \gamma(n) \}_{n=1}^\infty \) be an even sequence of non-negative numbers increasing to infinity. Let $\mu$ be a non-negative continuous, singular measure on the circle group $\mathbb{T}$ such that [13]

$$\hat{\mu}(a) \approx 0 \left( \frac{\gamma(n)}{n^{1/2}} \right) \text{ and } \mu(\mathbb{T}) = 1.$$ 

We define a compactly supported measure $\nu$ on $\mathbb{T}$ as follows: Let $f \in C(\mathbb{T}),$

$$\langle \nu, f \rangle = \int_0^{2\pi} \cdots \int_0^{2\pi} f(a_1, \ldots, a_\ell) \pi(a_1, \ldots, a_\ell) \mu(a_1) \cdots \mu(a_\ell)$$

where

$$f(a_1, \ldots, a_\ell) = f(H_p),$$

$$\pi(a_1, \ldots, a_\ell) = \prod_{i \in R_+} \hat{\alpha}(H_p) \text{ when } p = 2a_1 \hat{\alpha}_1 + \ldots + 2a_\ell \hat{\alpha}_\ell.$$ 

Let $\wedge = \{ H_p | \parallel \gamma(H_p) = 2a_1 \hat{\alpha}_1 + \ldots + 2a_\ell \hat{\alpha}_\ell, \ 0 \leq a_\ell \leq 2\pi, \ i = 1, \ldots, \ell \}$. Then $\nu$ is a singular measure on $\mathbb{T}$ supported in $\wedge$.

Next we define an $Ad$-invariant measure $\mu_g$ on $g$. Let $f \in C_0(g),$

$$\langle \mu_g, f \rangle = \int \int_{O_H} f(Y) d\beta(H) d\nu(H),$$

where $O_H$ denotes the orbit of $H$ in $g$ and $\beta(H)$ is the surface measure on $O_H$. Then $\mu_g$ is a continuous, singular, $Ad$-invariant measure of compact support on $g$.

Let $\mu_G = \Phi \mu_g$.

It is easily seen that $\mu_G$ is a continuous, central, singular measure on $G$. We show that $||\hat{\mu}_G||_p < \infty$ for $p > 2$. Let $\sigma \in G$; then $\hat{\mu}_G(\sigma) = \hat{\mu}_g(\lambda + \delta)I_d$, where $\lambda$ is the highest weight for the representation $\sigma$.

Now

$$\hat{\mu}_g(\lambda + \delta) = \int \left( \int_{O_H} e^{-i(\lambda + \delta)(X)} d\beta(H) \right) d\nu(H).$$

Using Harish-Chandra’s formula for the Fourier-Stieltjes transform of a surface measure [1], we have

$$\hat{\mu}_g(\lambda + \delta) = \frac{1}{\pi(H_{\lambda + \delta})} \int \frac{1}{\pi(H)} \sum_{w \in W} \sgn(w) e^{i(w, \lambda + \delta)(H)} d\nu(H)$$

$$= \frac{1}{\pi(H_{\lambda + \delta})} \sum_{w \in W} \sgn(w) \int_0^{2\pi} \cdots \int_0^{2\pi} e^{i(w(H), 2a_1 \hat{\alpha}_1 + \ldots + 2a_\ell \hat{\alpha}_\ell)} d\mu(a_1) \cdots d\mu(a_\ell).$$

$$= \frac{1}{\pi(H_{\lambda + \delta})} \sum_{w \in W} \sgn(w) \prod_{i=1}^\ell \hat{\mu} \left( 2(\lambda + \delta, w^{-1} \hat{\alpha}_i) \right).$$

Hence

$$||\hat{\mu}_G||_p^p = \sum_{\sigma \in G} d_\sigma ||\hat{\mu}_G(\sigma)||_p^p = \sum_{\sigma \in G} d_\sigma^2 |\hat{\mu}_g(\lambda + \delta)|^p.$$
By Weyl’s dimension formula, \( d_{\sigma} = \prod_{\alpha \in R_+} \langle \lambda + \delta, \hat{\alpha} \rangle \), thus we obtain

\[
\| \hat{\mu}_G \|_p^p \leq C \sum_{\sigma \in G} \left( \frac{1}{\prod_{\alpha \in R_+} \langle \lambda + \delta, \hat{\alpha} \rangle} \right)^{p-2} \left| \sum_{w \in W} \text{sgn}(w) a_w \right|^p
\]

where

\[
a_w = \frac{\prod_{i=1}^{\ell} \gamma(2\langle \lambda + \delta, w^{-1} \hat{\alpha}_i \rangle)}{\prod_{i=1}^{\ell} (2\langle \lambda + \delta, w^{-1} \hat{\alpha}_i \rangle)^{1/2}}, \quad w \in W.
\]

Since \( G \) is simply connected, \( \hat{G} \) is in one to one correspondence with the dominant integral weights given by

\[
\lambda = \sum_{i=1}^{\ell} m_i \lambda_i, \quad m_i \in \mathbb{Z}^+.
\]

Here, \( \{\lambda_1, \ldots, \lambda_\ell\} \subseteq \mathfrak{h}^* \) is the set of fundamental dominant weights, i.e. \( \langle \lambda_i, \hat{\alpha}_j \rangle = \delta_{ij} \). Fix \( \lambda \in \hat{G} \). Suppose \( \lambda = \sum_{i=1}^{\ell} m_i \lambda_i, \quad m_i \in \mathbb{Z}^+ \) and \( m = \max_{\alpha \in R} 2 |\langle \lambda + \delta, \hat{\alpha} \rangle| \).

Then \( m \leq C_1 (m_1 + \ldots + m_\ell) \). Also for \( w \in W \), \( \prod_{i=1}^{\ell} 2 |\langle \lambda + \delta, w^{-1} \hat{\alpha}_i \rangle| \geq \prod_{i=1}^{\ell} (m_i + 1) \). Hence

\[
\| \hat{\mu}_G \|_p^p \leq C \sum_{m_1, \ldots, m_\ell \in \mathbb{Z}^+} \frac{(\gamma(m))^{p\ell}|W|^p}{\prod_{i=1}^{\ell} (m_i + 1)^{p/2}} \prod_{i=1}^{\ell} (m_i + 1)^{p/2}
\]

\[
\leq C \sum_{m_1, \ldots, m_\ell \in \mathbb{Z}^+} \frac{(\gamma(m))^{p\ell}|W|^p}{\prod_{i=1}^{\ell} (m_i + 1)^{\frac{3p-2}{2}}}.
\]

Let

\[
\gamma(n) = 0 \quad \text{if } |n| \leq 1 \quad \text{and} \quad = \log |n| \quad \text{if } |n| > 1.
\]

Then

\[
\| \hat{\mu}_G \|_p^p \leq C |W|^p \sum_{m_1, \ldots, m_\ell \in \mathbb{Z}^+} \frac{(\log C_1 + \prod_{i=1}^{\ell} \log (m_i + 1))^{p\ell}}{\prod_{i=1}^{\ell} (m_i + 1)^{\frac{3p-2}{2}}}
\]

\[
\leq C_2 \sum_{m_1, \ldots, m_\ell \in \mathbb{Z}^+} \frac{(\prod_{i=1}^{\ell} \log (m_i + 1))^{p\ell}}{\prod_{i=1}^{\ell} (m_i + 1)^{\frac{3p-2}{2}}}
\]

\[
= C_2 \prod_{i=1}^{\ell} \sum_{m=0}^{\infty} \frac{(\log (m + 1))^{p\ell}}{(m + 1)^{\frac{3p-2}{2}}} < \infty \text{ for } p > 2.
\]

The proof is complete.

To complete the proof of Theorem 3.1 in the general case, we first prove two lemmas. For convenience, we will say that the group \( G \) has property \( P \) if there exists a continuous, singular, central measure on \( G \) whose Fourier-Stieltjes transform is \( p \)-summable for every \( p > 2 \).
Lemma 3.2. Let $T$ be a compact connected abelian group, $\{G_\alpha\}_{\alpha \in I}$ a family of compact, simply connected, simple Lie groups and $K$ a totally disconnected closed subgroup of the centre of $T \times \prod_{\alpha \in I} G_\alpha$. Suppose that there exists a $\beta \in I$ such that $G_\beta$ has property $P$. Then

$$G = T \times \prod_{\alpha \in I} G_\alpha/K$$

has property $P$.

Proof. Let $\mu_\beta$ be a measure on $G_\beta$ corresponding to property $P$ of $G_\beta$. Define $\nu = \prod_{\alpha \in (I - \{\beta\})} \mu_\alpha \times \mu_\beta$, where $\mu_\alpha$ is Haar measure on $G_\alpha$ for $\alpha \in (I - \{\beta\})$.

Then $\nu$ is a continuous, central, singular measure on $T \times \prod_{\alpha \in I} G_\alpha$ and the measure $\mu$ induced by $\nu$ on $G$ shows that $G$ has property $P$. This completes the proof.

Lemma 3.3. Let $G$ be any compact group. Let $G_0$ be an open normal subgroup of $G$ satisfying the property $P$. Then $G$ satisfies $P$.

Proof. Define a measure $\mu$ on $G$ as follows: Let $E$ be a Borel measurable subset of $G$,

$$\mu(E) = \mu_0(E \cap G_0),$$

when $\mu_0$ is a measure on $G_0$ satisfying $P$. It is clear that $\mu$ is central, singular, continuous measure on $G$. We show that $||\hat{\mu}||_p < \infty$ for every $p > 2$.

Let $n$ be the index of $G_0$ in $G$. If $\sigma \in \hat{G}$, then by Clifford's theorem ([8], Theorem 12.1) there exists $\alpha \in \hat{G}_0$ and natural numbers $k, n_\sigma (k, n_\sigma \leq n)$ such that

$$\sigma|_{\hat{G}_0} = n_\sigma (\alpha(g_1) + \ldots + \alpha(g_k))$$

when

$$\alpha^{(g)}(h) = \alpha(g^{-1}h), \quad g \in G, h \in G_0,$$

and $\{g = g_1, g_2, \ldots, g_k\}$ is a complete set of coset representatives for $H_\alpha = \{g \in G|\alpha(g)\}$ is equivalent to $\alpha_1$.

Let $\hat{\mu}(\sigma) = c_\sigma I_{d_\sigma}$. Now

$$c_\sigma = \frac{1}{d_\sigma} \int_G \chi_\sigma(x^{-1})d\mu(x)$$

$$= \frac{n_\sigma}{d_\sigma} (c_{\alpha(g_1)} + \ldots + c_{\alpha(g_k)})$$

where $\hat{\mu}_0(\alpha(g_i)) = c_{\alpha(g_i)} I_{d_\sigma}$. Let $p > 2$. Then

$$||\hat{\mu}||_p = \sum_{\sigma \in \hat{G}} d_\sigma ||\hat{\mu}(\sigma)||_p = \sum_{\sigma \in \hat{G}} d_\sigma^2 |c_\sigma|^p$$

$$= \sum_{\sigma \in \hat{G}} d_\sigma^2 \left(\frac{n_\sigma}{d_\sigma} \right)^p \left(\sum_{i=1}^k |c_{\alpha(g_i)}|\right)^p.$$
Since ‘α’ can occur in at most n irreducible representations of G (as 1 ≤ n ≤ n), we get
\[ \|\hat{\mu}\|_p^p \leq n \sum_{\alpha \in \hat{G}} d_\alpha^2 n_\sigma \left( \sum_{i=1}^k |c_{\alpha(s_i)}| \right)^p \]
\[ \leq n^4 \sum_{\alpha \in \hat{G}_0} d_\alpha^2 \max_{1 \leq i \leq k} |c_{\alpha(s_i)}|^p. \]
Now \( \sum_{\alpha \in \hat{G}_0} d_\alpha^2 \max_{1 \leq i \leq k} |c_{\alpha(s_i)}|^p \leq \sum_{\alpha \in \hat{G}_0} d_\alpha^2 |c_\alpha|^p. \) Therefore
\[ \|\hat{\mu}\|_p^p \leq n^4 \sum_{\alpha \in \hat{G}_0} d_\alpha^2 |c_\alpha|^p = n^4 \|\hat{\mu}_0\|_p^p < \infty \]
for \( p > 2. \)

This completes the proof.

End of the proof of Theorem 3.1. The theorem follows now by a standard argument using the structure theorem of compact connected groups (see [9]), Lemma 3.2 and Lemma 3.3.

Remark 3.4. S. Sakai [12] has shown that there exists a singular probability measure \( \mu \) on the circle group \( \mathbb{T} \) such that \( \mu \ast \mu \) is absolutely continuous and the Radon-Nikodym derivative of \( \mu \ast \mu \) has a uniformly convergent Fourier series. We are unable to show that the measure \( \mu \) constructed in Theorem 3.1 has this stronger property.

4. Totally disconnected groups

In this section we prove the following analogue of Theorem 3.1 on some totally disconnected groups.

Theorem 4.1. Let \( \{G_n\}_{n=1}^\infty \) be a sequence of finite non-abelian groups such that there exist two constants \( C_1, C_2 > 0 \) satisfying: For every \( n \), there exists an irreducible character \( \chi_n \) of \( G_n \) such that
\[ \text{Re} (\chi_n) \geq -C_1 \quad \text{on } G_n \]
\[ \frac{1}{|G_n|} \sum_{x \in A_n} (\text{Re} \chi_n(x))^2 \geq C_2 \]
when \( A_n = \{ x \in G_n \mid \text{Re} \chi_n(x) < 0 \}. \)

Then there exists a continuous, singular, central measure \( \mu \) on \( G = \prod_{n=1}^\infty G_n \) such that \( \mu \ast \mu \) is absolutely continuous with respect to Haar measure on \( G \) and the Lebesgue Radon-Nikodym derivative of \( \mu \ast \mu \) belongs to \( L^p \) for all \( p, 1 \leq p < \infty. \)

Proof. Let \( \{b_n\}_{n=1}^\infty \) be a sequence of non-negative numbers such that
\[ 0 < b_n C_1 < 1 \quad \text{for every } n, \quad \sum_{n=1}^\infty b_n^2 = \infty \quad \text{and} \quad \sum_{n=1}^\infty b_n^p < \infty, \quad \text{for every } p > 2. \]

Define \( g_n = \text{Re} \chi_n \), and \( f_n = 1 + b_n g_n. \)
Let \( \mu = \prod_{n=1}^{\infty} f_n dx \), where \( dx \) denotes the Haar measure on \( G_n \). Then \( \mu \) is a central continuous measure on \( G \).

\( \mu \) is singular: By Kakutani’s criterion ([5], Theorem 22.36) for singularity of product measures, it is sufficient to show that

\[
\prod_{n=1}^{\infty} \int_{G_n} f_n^{1/2} dx = 0.
\]

Now

\[
\int_{G_n} f_n^{1/2} dx = \int_{A_n} f_n^{1/2} dx + \int_{G_n - A_n} f_n^{1/2} dx.
\]

Also

\[
\int_{A_n} f_n^{1/2} dx = \frac{1}{|G_n|} \sum_{x \in A_n} (1 - (-b_n g_n(x)))^{1/2} \leq \frac{1}{|G_n|} \sum_{x \in A_n} \left( 1 + \frac{b_n g_n(x)}{2} - \frac{b_n^2}{16} g_n^2(x) \right),
\]

and

\[
\int_{G_n - A_n} f_n^{1/2} dx \leq \left( \int_{G_n - A_n} f_n dx \right)^{1/2} \left( \frac{|G_n| - |A_n|}{|G_n|} \right)^{1/2} \leq \frac{|G_n| - |A_n|}{2(|G_n| - |A_n|)} \left( 1 + \sum_{x \in A_n} (-b_n g_n(x)) \right).
\]

By (4.4) and (4.5), we get

\[
\int_{G_n} f_n^{1/2} dx \leq 1 - \frac{1}{|G_n|} \sum_{x \in A_n} \frac{b_n^2 g_n^2(x)}{16} \leq 1 - \frac{b_n^2 C_2}{16}.
\]

Since \( \sum_{n=1}^{\infty} b_n^2 = \infty \), we conclude ([5], Lemma 22.25) that \( \prod_{n=1}^{\infty} \int_{G_n} f_n^{1/2} = 0 \).

Computation of \( \|\hat{\mu}\|_p \): Let \( p > 2 \).

\[
\|\hat{\mu}\|_p = \sum_{\sigma \in \mathcal{G}} \|\hat{\mu}(\sigma)\|_p^p = \sum_{\sigma \in \mathcal{G}} d_\sigma^p \prod_{i=1}^{k} \left( \frac{b_n}{2} \left( 1 + \int_{G_n} \chi_n(x)^2 dx \right) \right)^p \leq \sum_{n_1, \ldots, n_k \in \mathbb{N}} \prod_{i=1}^{k} (b_{n_i}^p) = \prod_{i=1}^{k} (1 + b_i^p) < \infty \text{ as } \sum_{i=1}^{\infty} b_i^p < \infty.
\]

This completes the proof of the theorem.

Remark 1. Conditions (4.2) and (4.3) are satisfied by \( G = \prod_{n=1}^{\infty} G_n \) if \( G_n \)'s are symmetric groups, dihedral groups and generalised quaternion groups (see [4], 27.61).

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