A COUNTEREXAMPLE TO CARTAN’S CONJECTURE
ON HOLOMORPHIC CURVES OMITTING HYPERPLANES

ALEXANDRE EREMENKO

(Communicated by Albert Baernstein II)

Abstract. In his 1928 thesis H. Cartan proved a theorem which can be considered as an extension of Montel’s normality criterion to holomorphic curves in complex projective plane \( \mathbb{P}^2 \). He also conjectured that a similar result is true for holomorphic curves in \( \mathbb{P}^n \) for any \( n \). A counterexample to this conjecture is constructed for any \( n \geq 3 \).

The following theorem of Borel may be considered as an extension of Picard’s theorem to holomorphic mappings of the complex plane \( \mathbb{C} \) to complex projective space.

Borel’s Theorem. Let \( f_1, \ldots, f_p \) be a system of entire functions without zeros and

\[
 f_1 + \ldots + f_p = 0.
\]

Then the set of indices \( \{1, \ldots, p\} \) can be partitioned into disjoint subsets \( \{I\} \) such that \( |I| \geq 2 \), and for every \( I \) the functions \( f_j, j \in I \), are proportional and their sum is zero.

According to the so-called Bloch principle, to every theorem of Picard type should correspond a Montel-type theorem for families of functions in the unit disk. The following statement is known as

Cartan’s Conjecture ([2, 3]). Let \( \mathcal{F} \) be an infinite family of \( p \)-tuples of holomorphic functions \( f = (f_1, \ldots, f_p) \) without zeros in the unit disk \( U \) satisfying the Borel equation (1).

Then there exists an infinite subsequence \( \mathcal{L} \) having the following property.

There exists a partition of indices \( P = \{1, \ldots, p\} \) into disjoint sets \( \{S\} \) and each \( S \) contains a subset \( I \) with at least two elements, which may be equal to \( S \) itself. These satisfy the following properties for \( f \in \mathcal{L} \):

(i) For each \( S \) and \( j, k \in I \subset S \) the sequence \( \{f_j/f_k\} \) is convergent (uniformly on compacta, to a non-zero function).

(ii) If \( j \in S \setminus I \) and \( k \in I \subset S \) then \( f_j/f_k \) converges to 0.

(iii) Given \( k \in I \subset S \),

\[
 \sum_{j \in I} f_j/f_k \text{ converges to } 0.
\]
When $p = 3$ the statement is (almost) equivalent to the Montel theorem, which asserts that a family of meromorphic functions in the unit disk omitting three given values is normal. Cartan [2], see also [3, Ch. VIII], proved a partial result:

Let $\mathcal{F}$ be as above. Then there exists a subsequence $\mathcal{L} \subset \mathcal{F}$ having one of the following properties:

(a) The full set $P$ of indices satisfies (i), (ii) and (iii) (with single set $S = P$), or

(b) There are two disjoint subsets $S_1$ and $S_2$ in $P$, each containing at least two elements, satisfying the three conditions (i), (ii) and (iii).

The point is that $S_1$ and $S_2$ in (b) may not cover the whole set of indices $P$. This result implies that Cartan's conjecture is true for $p = 3$ and $p = 4$ [2]. We show that it fails for $p = 5$.

**Example.** It is convenient to work in the rectangle $R = \{x + iy : |x| < \pi, 0 < y < 1\}$ instead of the unit disk. For every natural integer $n > 12 > 4e$ consider the function $h(z) = h_n(z) = \exp(n \exp iz)$, $z \in R$. We have

$$
\log |h_n(x + iy)| = n \cos x \exp(-y).
$$

The set $\{z \in R : |h_n(z)| < 3\}$ consists of two components: left and right. We denote the right component by $D_n$ so that as $n \to \infty$, $D_n \to R \cap \{x \geq \pi/2\}$. Choose a diffeomorphism $p$ of the disk $\{w : |w| \leq 3\}$ onto itself with the following properties:

$$
p(w) = w, \quad |w| = 3,
$$

$$
p(0) = 1
$$

and

$$
p \text{ is conformal for } |w| < 2.
$$

Put

$$
\hat{G}_n(z) = \begin{cases} 
p \circ h_n(z), & z \in D_n, \\
h_n(z), & z \in R\setminus D_n.
\end{cases}
$$

Then we can find a diffeomorphism $\phi_n : R \to R$, continuous in $\bar{R}$ with

$$
\phi_n(0) = 0, \quad \phi_n(\pm \pi) = \pm \pi
$$

such that

$$
G_n = \hat{G}_n \circ \phi_n^{-1}
$$

is holomorphic in $R$. This $\phi_n$ is obtained by solving a Beltrami equation [1]

$$
\frac{\partial \phi_n}{\partial \bar{z}} = \mu \frac{\partial \phi_n}{\partial z},
$$

where $\mu$ is a smooth function, $|\mu(z)| \leq c \leq 1$, $z \in R$, $c$ an absolute constant, and

$$
supp \mu = K_n = \{z \in R : \Re z > 0, 2 \leq |h_n(z)| \leq 3\}.
$$

We claim that

$$
\phi_n(z) - z \to 0, \quad n \to \infty
$$

uniformly on $R$. Indeed, $\{\phi_n\}$ is a family of quasiconformal homeomorphisms of $R$ with uniformly bounded dilatation, so this family is precompact (the topology of
uniform convergence). Any limit function $\phi$ of the family is conformal everywhere in $R$ except perhaps the segment

$$K = \{ \pi/2 + it : 0 < t < 1 \} = \lim_{n \to \infty} K_n.$$  

But $K$ is a removable singularity for homeomorphisms conformal in the complement of $K$. So $\phi$ is a conformal automorphism of $R$ and (2) implies that $\phi = \text{id}$. This proves (4). Notice that $G_n - 1$ has no zeros in $R \cap \{ x > 0 \}$ and $G_n$ has no zeros in $R \cap \{ x < 0 \}$. It follows from (4) that

$$\log |G_n(x + iy) - 1| = (n + o(1)) \cos x \exp(-y), \quad x > 0$$

(5)

and

$$\log |G_n(x + iy)| = (n + o(1)) \cos x \exp(-y), \quad x < 0,$$

when $n \to \infty$ uniformly on $R$. Now we define $H_n$ by

$$G_n + H_n = 1.$$  

(7)

Asymptotic equalities (5) and (6) imply respectively

$$\log |H_n(x + iy)| = (n + o(1)) \cos x \exp(-y), \quad x > 0$$

(8)

and

$$\log |H_n(x + iy) - 1| = (n + o(1)) \cos x \exp(-y), \quad x < 0,$$

(9)

as $n \to \infty$ uniformly on $R$.

Now we set $a = \pi - 1/(e + 1)$ and define

$$f_1^1(z) = \exp\{n(z + a)\}, \quad f_2^1(z) = \exp\{n(-z + a)\},$$

$$f_3^1 = G_n - f_1^1, \quad f_4^1 = H_n - f_2^1, \quad f_5^1(z) \equiv 1.$$  

From this definition and (7) follows that (1) is satisfied. Furthermore we have in view of (5), (6), (8) and (9)

$$|G_n| < |f_1^1| \quad \text{and} \quad |H_n| < |f_2^1| \quad \text{in} \quad R,$$

(10)

for $n$ large enough.

Inequalities (10) show that all five functions $f_j^1$ are zero-free in $R$ if $n$ is large enough.

Now we show that the conclusion of Cartan’s conjecture is not valid for the functions of our sequence. This is because $f_5^1$ cannot be in the same class $S$ with any other function $f_j^1$, $1 \leq j \leq 4$. Indeed, when $j$ is odd we have

$$\log |f_j^1(z)| = (n + o(1))(\Re z + a), \quad n \to \infty,$$

so

$$f_j^1\left(-\pi + \frac{1}{2(e + 1)} + \frac{i}{2}\right) \to 0 \quad \text{and} \quad f_j^1(i/2) \to \infty, \quad n \to \infty.$$  

A similar argument works for even $j$. In this case

$$f_j^1\left(\pi - \frac{1}{2(e + 1)} + \frac{i}{2}\right) \to 0 \quad \text{and} \quad f_j^1(i/2) \to \infty, \quad n \to \infty.$$  

So $f_5^1 \equiv -1$ cannot be included in any class $S$ described in (i) and (ii) of Cartan’s conjecture.
Remarks. The simplest counterexample for any \( p > 6 \) can be constructed by adding non-zero constant functions \( f^j_n \) with the properties

\[
\sum_{j=6}^{p} f^j_n = 0
\]

and \( |f^j_n| = b^{-n}, \ 6 \leq j \leq p \), where \( 1 < b < \exp\{1/(e+1)\} \). These new functions may be included in one class \( S \) with \( f^5_n \) but then (iii) fails for this class. Our example for \( p = 5 \) shows that even a partition into classes \( S, \text{card} \ S \geq 2, \) which satisfy (i) and (ii), is impossible. Examples with this property can also be constructed for any \( p > 5 \).

The author thanks David Drasin, who made many helpful suggestions, and V. Lin for illuminating discussions.

References


Department of Mathematics, Purdue University, West Lafayette, Indiana 47907

E-mail address: eremenko@math.purdue.edu