EQUIVALENT CONDITIONS INVOLVING COMMON FIXED POINTS FOR MAPS ON THE UNIT INTERVAL

JACEK R. JACHYMSKI

(Communicated by James E. West)

Abstract. Let $g$ be a continuous self-map of the unit interval $I$. Equivalent conditions are given to ensure that $g$ has a common fixed point with every continuous map $f : I \to I$ that commutes with $g$ on a suitable subset of $I$. This extends a recent result of Gerald Jungck.

1. Introduction

Let $f$ and $g$ be two commuting continuous self-maps of $I$, the unit interval. It is known that $f$ and $g$ need not have then a common fixed point (for counterexamples, see, e.g., [6]). However, if one of the maps, say $g$, has appropriate additional properties then $f$ and $g$ possess a common fixed point. In particular, W. Boyce [1, Corollary 5] has shown that it suffices to assume the family $\{g^n : n \in \mathbb{N}\}$, iterates of $g$, is equicontinuous on $I$. This result has been extended by J. Cano [2, Theorems 1 and 2] who has required that either $g$ has a closed interval for its set of fixed points $F(g)$, or $F(g)$ coincides with $P(g)$, the set of all periodic points of $g$. It is worth underlining here that Corollary 5 [1] as well as Theorem 1 [2] give only sufficient conditions for the existence of a common fixed point of $f$ and $g$.

On the other hand, recently, Gerald Jungck [7, Theorem 3.6] established the following interesting equivalence: a continuous self-map $g$ of $I$ has a common fixed point with every continuous map $f : I \to I$ that nontrivially commutes with $g$ on the set of coincidence points of $f$ and $g$ if and only if $P(g) = F(g)$.

Our purpose here is to give other necessary and sufficient criteria of this type (see Theorems 1, 2 and 3). We also obtain a variant of Jungck’s Theorem in more abstract settings as compact and convex subsets of a normed linear space (see Proposition 1).

2. Equivalent conditions

Following Boyce [1] and Cano [2] we define the classes of maps:

$$B \triangleq \{g : I \to I \mid \{g^n : n \in \mathbb{N}\} \text{ is equicontinuous on } I\},$$

$$C_1 \triangleq \{g : I \to I \mid g \text{ is continuous and } F(g) \text{ is a closed interval}\},$$

$$C_2 \triangleq \{g : I \to I \mid g \text{ is continuous and } F(g) = P(g)\}.$$
Let $f$ and $g$ be continuous self-maps of $I$. Then we have:

1. If $f$ and $g$ commute on $I$ and $g \in B$ then $F(f) \cap F(g) \neq \emptyset$ [1, Corollary 5].
2. If $f$ and $g$ commute on $I$ and $g \in C_1 \cup C_2$ then $F(f) \cap F(g) \neq \emptyset$; moreover, $B \subseteq C_1$. If $g \in B$ and $F(g)$ is not a singleton then $g \in C_2$ [2, Theorems 1 and 2].

Next, by [7, Theorem 3.6], $g \in C_2$ if and only if $F(f) \cap F(g) \neq \emptyset$ for every continuous map $f : I \to I$ such that the set of coincidence points of $f$ and $g$ is non-empty, and $f$ and $g$ commute on it.

Inspired by the last result we give other characterizations of the classes $C_1, C_2$ and $B$.

**Theorem 1.** Let $g$ be a continuous self-map of $I$. Then the following conditions are equivalent:

1. $g \in C_1$;
2. the family $\{g^n : n \in N\}$ is equicontinuous on $F(g)$, or $F(g)$ is a singleton;
3. $g$ has a common fixed point with every continuous map $f : I \to I$ that commutes with $g$ on $F(g)$.

**Proof.** (i) $\Rightarrow$ (ii). If $F(g)$ is a singleton, we are done. So suppose $F(g) = [a, b]$ and $a < b$. Obviously, it suffices to show that $\{g^n : n \in N\}$ is equicontinuous at the points $a$ and $b$. And because of symmetry, we consider only the point $a$. The case when $a = 0$ is trivial. So suppose $a > 0$. Fix an $\epsilon \in (0, b - a)$. By the continuity, there exists a $\delta \in (0, \epsilon)$ such that

\[ a - \epsilon < g(x) < a + \epsilon \text{ for all } x \in (a - \delta, a) \cap I. \]

We shall apply induction on $n$ to show that, for all $n \in N$,

\[ a - \epsilon < g^n(x) < a + \epsilon \text{ for all } x \in (a - \delta, a) \cap I. \]

By (1), (2) holds if $n = 1$. Assuming (2) holds for $n = 1, 2, \ldots, k$, we shall prove it for $k + 1$. Fix an $x \in (a - \delta, a) \cap I$. If $a \leq g^k(x) < a + \epsilon$ then $g^{k+1}(x) \in F(g)$ since $a + \epsilon < b$ so (2) is fulfilled for $n = k + 1$. Assume now that $g^k(x) < a$. Then $g^i(x) < a$ for $i = 1, 2, \ldots, k$; for otherwise, by induction hypothesis, $a \leq g^i(x) < a + \epsilon$ for some $i$, $1 \leq i \leq k$ so $g^i(x) \in F(g)$, which implies $g^{k+1}(x) \in F(g)$ and hence $g^{k+1}(x) \geq a$, a contradiction. Since $F(g) = [a, b]$, we have $g(a) > y$ for all $y \in [0, a)$. In particular, $g^i(x) > g^{i-1}(x)$ for $i = 1, 2, \ldots, k$, which implies $g^k(x) > x$. Since $x > a - \delta$ and $g^k(x) < a$, we obtain that $g^k(x) \in (a - \delta, a) \cap I$. By (1), $a - \epsilon < g^{k+1}(x) < a + \epsilon$, which completes the induction.

Since $\delta < \epsilon < b - a$, we have $g^n(x) = x$ for $x \in [a, a + \delta)$ and $n \in N$. So finally, $a - \epsilon < g^n(x) < a + \epsilon$ for all $x \in (a - \delta, a + \delta)$ and $n \in N$. This proves $\{g^n : n \in N\}$ is equicontinuous at the point $a$.

(ii) $\Rightarrow$ (i). This implication follows from the proof of Cano’s Theorem 1 [2].

(i) $\Rightarrow$ (iii). If $f$ commutes with $g$ on $F(g)$ then $F(f)$ is $f$-invariant so $f|_{F(g)}$ has a fixed point since $F(g)$ is a closed interval.

(iii) $\Rightarrow$ (i). Suppose $F(g)$ is not an interval. There exist $a, b \in F(g)$, $a < b$, such that $(a, b) \cap F(g) = \emptyset$. Define the map $f : f(x) = b$ for $x \in [0, a]$, $f(x) = -x + a + b$ for $x \in (a, b)$, and $f(x) = a$ for $x \in [b, 1]$. Then $f$ is continuous and for $x \in F(g)$, either $x \in [0, a]$ and then $f(g(x)) = g(f(x)) = b$, or $x \in [b, 1]$ and then $f(g(x)) = g(f(x)) = a$. Thus $f$ and $g$ commute on $F(g)$, but $F(f) \cap F(g) = \emptyset$, which contradicts (iii).
The following example shows that we cannot omit the condition “$F(g)$ is a singleton” in (ii) of Theorem 1.

**Example 1.** Define the map $g$ on $I$ as follows:

$$g(x) = \begin{cases} 1 & \text{for } x \in [0, \frac{1}{4}], \\ -2x + \frac{3}{2} & \text{for } x \in \left(\frac{1}{4}, \frac{3}{4}\right), \\ 0 & \text{for } x \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

Clearly, $F(g) = \{\frac{1}{2}\}$ and $g \in C_1$. However, it is easy to verify that the family $\{g^n : n \in \mathbb{N}\}$ is not equicontinuous at the point $\frac{1}{2}$.

**Theorem 2.** Let $g$ be a continuous self-map of $I$. Then the following conditions are equivalent:

(i) $g \in C_2$;

(ii) the sequence $\{g^n\}_{n=1}^{\infty}$ is pointwise convergent on $I$;

(iii) $g$ has a common fixed point with every continuous map $f : I \to I$ that commutes with $g$ on $F(f)$.

**Proof.** That (i) implies (ii) was proved by S. C. Chu and R. D. Moyer [3, Theorem 1] and, independently, by E. M. Coven and G. A. Hedlund [4, Theorem 2]. To prove (ii) implies (iii) fix an $x \in F(f)$. Since, by the commutativity, $F(f)$ is $g$-invariant we have $g^n(x) \in F(f)$ for $n \in \mathbb{N}$. By (ii), $\{g^n(x)\}_{n=1}^{\infty}$ converges to some $z \in I$. Then $z \in F(f) \cap F(g)$ since $F(f)$ is closed and $g$ is continuous. To prove (iii) implies (i) it suffices to show that for any non-empty closed $g$-invariant set $C \subseteq I$, $C \cap F(g) \neq \emptyset$ and then apply [3, Theorem 1]. Fix such a set $C$. There exists a continuous map $f : I \to I$ such that $F(f) = C$. If $x \in F(f)$ then $g(f(x)) = g(x)$ and $f(g(x)) = g(x)$ since $C$ is $g$-invariant. Thus $f$ and $g$ commute on $F(f)$ so, by (iii), $F(f) \cap F(g) \neq \emptyset$, i.e., $C \cap F(g) \neq \emptyset$.

**Remark 1.** The sufficiency part of Jungck’s Theorem 3.6 [7] is easily subsumed by Theorem 2: if $f$ and $g$ commute at their coincidence points, $P(g) = F(g)$ and $f(a) = g(a)$ then $f^n(a) = g^n(a)$ for $n \in \mathbb{N}$. By Theorem 2 ((i) ⇒ (ii)), $\{g^n(a)\}_{n=1}^{\infty}$ is convergent to some $b$. Then, by the continuity, $b$ is a common fixed point of $f$ and $g$.

Before stating the next theorem let us notice that a common fixed point theorem for a family of commuting maps would be trivial if we assumed one of them had a unique fixed point. This justifies a use of the assumption “$F(g)$ is not a singleton” in Theorem 3 below.

**Theorem 3.** Let $g$ be a continuous self-map of $I$ such that $F(g)$ is not a singleton. Then the following conditions are equivalent:

(i) $g \in B$;

(ii) the sequence $\{g^n\}_{n=1}^{\infty}$ is uniformly convergent on $I$;

(iii) $g$ has a common fixed point with every continuous map $f : I \to I$ that commutes with $g$ either on $F(f)$, or on $F(g)$.

**Proof.** (i) ⇒ (ii). If $g \in B$ and $F(g)$ is not a singleton then, by [2, Theorem 2], $g \in C_2$. By Theorem 2, $\{g^n\}_{n=1}^{\infty}$ is pointwise convergent on $I$ which implies (ii), since $\{g^n : n \in \mathbb{N}\}$ is equicontinuous.

(ii) ⇒ (iii). This implication easily follows from Theorems 1 and 2.

(iii) ⇒ (i). By Theorem 2, $\{g^n\}_{n=1}^{\infty}$ is pointwise convergent, which implies $F(g) = F(g^2)$. On the other hand, by Theorem 1, $F(g)$ is an interval. Therefore, $F(g^n)$ is
an interval, so by [1, Lemma 1 and Theorem 5], \( \{g^n : n \in \mathbb{N}\} \) is equicontinuous, i.e., \( g \in B \).

**Remark 2.** The condition that \( F(g) \) is not a singleton is necessary in Theorem 3 (consider the map \( g(x) = 1 - x \) \( x \in I \)), for which (i) holds but (ii) is not fulfilled. Further, Example 1 shows that one cannot modify Theorem 3 similarly changes of the text of this paper.

Remark 3. It follows from [3, Theorem 1] that in case when \( A = I \), the conditions (i) of Proposition 1 and (i) of Theorem 2 are equivalent.

**Proof of Proposition 1.** (i) \( \Rightarrow \) (ii). Let a continuous map \( f : A \to A \) commute with \( g \) on \( F(f) \). Then \( F(f) \) is \( g \)-invariant and closed. Moreover, by Schauder’s Fixed Point Theorem, \( F(f) \) is non-empty. So by (i), we get \( F(f) \cap F(g) \neq \emptyset \).

(ii) \( \Rightarrow \) (i). Let \( C \) be a non-empty closed \( g \)-invariant subset of \( A \). We show that \( C \cap F(g) \neq \emptyset \). The case when \( C = A \) is trivial. So assume \( C \neq A \) and fix a point \( a \in A \setminus C \). There exists a continuous function \( \phi : A \to I \) such that \( \phi^{-1}(0) = a \) and \( \phi^{-1}(1) = C \) (see, e.g., [5, Theorem 1.5.19, p.69]). Assume further that \( 0 \in C \). Define a map \( f \) by \( f(x) = \phi(x)x \) for \( x \in A \). Then \( f(A) \subseteq A \) by convexity, and \( F(f) = C \), since \( x = f(x) \) if \( x = 0 \) or \( \phi(x) = 1 \). For \( x \in C \), \( g(f(x)) = g(x) \) since \( f(x) = x \), and \( f(g(x)) = g(x) \) since \( C \) is \( g \)-invariant and \( F(f) = C \). Thus \( f \) and \( g \) commute on \( F(f) \). By (ii), \( F(f) \cap F(g) \neq \emptyset \), i.e., \( C \cap F(g) \neq \emptyset \). In case when \( 0 \notin C \), fix a point \( c \in C \), consider the sets \( A' = A - c \), \( C' = C - c \), and repeat the above argument to deduce there exists a continuous map \( f' : A' \to A' \) such that \( F(f') = C' \). Next, define a map \( f \) as \( f(x) = f'(x - c) + c \) for \( x \in A \) and apply (ii) for such a map \( f \) to obtain that \( C \cap F(g) \neq \emptyset \).

**Acknowledgement**

The author is grateful to the referee for suggesting some stylistic and expositional changes of the text of this paper.

**References**


Institute of Mathematics, Technical University of Łódź, Żwirki 36, 90-924 Łódź, Poland

E-mail address: jachymski@lodz1.p.lodz.pl