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BIDIMENSIONAL LINEAR SYSTEMS WITH SINGULAR DYNAMICS

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ABSTRACT. We analyze a class of bidimensional linear systems for which the following characteristics are generic: the system is recurrent and there exists a unique ergodic measure which is concentrated in one ergodic sheet. The trajectories exhibit an oscillatory behaviour from one to the other side of the ergodic sheet which assures the proximal character of the flow.

1. INTRODUCTION

Let (Ω, Ξ) be a minimal and uniquely ergodic real flow on a compact metric space Ω, and let the $\mathbb{R}$-action be denoted by $(\xi, t) \to \xi_t$ for each $\xi \in \Omega$, $t \in \mathbb{R}$. We will denote by m_0 the unique normalized ergodic measure on Ω.

Given a continuous function $A : \Omega \to L(\mathbb{R}^2)$ with $\operatorname{tr} A = 0$ we consider a family of bidimensional linear systems

$$\mathbf{z}' = \begin{pmatrix} a(\xi_t) & b(\xi_t) \\ c(\xi_t) & -a(\xi_t) \end{pmatrix} \mathbf{z} = A(\xi_t)\mathbf{z}\,, \quad \xi \in \Omega\,. \tag{1.1}$$

These equations induce a skew-product flow in the line bundle $V_{\mathbb{C}} = \Omega \times \mathbb{C}^2$ ($V_{\mathbb{R}} = \Omega \times \mathbb{R}^2$). It takes $(t, \xi, \mathbf{z}_0)$ to $(\xi_t, \mathbf{z}(t, \xi, \mathbf{z}_0))$, where $\mathbf{z}(t, \xi, \mathbf{z}_0)$ satisfies the equation defined by (1.1) along the trajectory passing through ξ with the initial data $\mathbf{z}(0, \xi, \mathbf{z}_0) = \mathbf{z}_0$. By linearity on the fibers the map $\Pi : V_{\mathbb{C}} \to K_{\mathbb{C}} = \Omega \times P^1(\mathbb{C})$, $(\xi, \mathbf{z}) \to (\xi, z_2/z_1)$, transports this flow to the projective bundle. The symbol Φ_A will represent the flow application in any invariant subset of $K_{\mathbb{C}}$ that is considered.

Taking the complex coordinate $Z = z_2/z_1$ in (1.1) we obtain the Riccati equations

$$Z' = c(\xi_t) - 2a(\xi_t)Z - b(\xi_t)Z^2\,. \tag{1.2}$$

If we denote by $Z(t, \xi, Z_0)$ the solution of the equation (1.2) with the initial data $Z(0, \xi, Z_0) = Z_0$, then $\Phi_A(t, \xi, Z_0) = (\xi_t, Z(t, \xi, Z_0))$ defines the equation of the flow on $K_{\mathbb{C}}$.

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It is obvious that $K_{\mathbb{R}} = \Omega \times P^1(\mathbb{R})$ is a closed and invariant subset of $K_{\mathbb{C}}$; thus we can consider Φ_A acting on $K_{\mathbb{R}}$. We can identify $P^1(\mathbb{R})$ with the quotient space $\mathbb{R}/\pi\mathbb{Z}$. Writing the real solutions of (1.1) in polar symplectic coordinates

$$\varphi = \cot^{-1}(x_2/x_1), \quad \rho = (x_1^2 + x_2^2)/2$$

we obtain

$$\begin{aligned} \varphi' &= f(\xi_t, \varphi) = b(\xi_t)\cos^2\varphi - c(\xi_t)\sin^2\varphi + 2a(\xi_t)\sin\varphi\cos\varphi \\ &= \frac{1}{2}(b(\xi_t) - c(\xi_t)) + a(\xi_t)\sin 2\varphi + \frac{1}{2}(b(\xi_t) + c(\xi_t))\cos 2\varphi, \\ \rho' &= -\frac{\partial f}{\partial \varphi}(\xi_t, \varphi)\rho = (-2a(\xi_t)\cos 2\varphi + (b(\xi_t) + c(\xi_t))\sin 2\varphi)\rho\,. \end{aligned} \tag{1.3}$$

We denote by $\varphi(t, \xi, \varphi_0)$ both the solution of (1.3) as a function on $\mathbb{R}$ or on $P^1(\mathbb{R})$ with initial condition $\varphi(0, \xi, \varphi_0) = \varphi_0$: the context will give in each case the exact meaning of the symbol. The mapping $\Phi_A(t, \xi, \varphi_0) = (\xi_t, \varphi(t, \xi, \varphi_0))$ is precisely the restriction of the flow to $K_{\mathbb{R}}$. The relation $X = \cot\varphi$ gives us the change between the systems of coordinates that we have introduced. We will denote by r_1 the normalized Lebesgue measure on $P^1(\mathbb{R})$ and by $m_1 = m_0 \otimes r_1$ the product measure on $K_{\mathbb{R}}$.

We call the family of systems (1.1) *recurrent-proximal* if the skew-product flow $(K_{\mathbb{R}}, \Phi_A)$ is recurrent and is a proximal extension of the flow (Ω, Ξ). For easy reference we recall that a flow is called *recurrent* or *minimal* if the orbit of every point is dense, and $(K_{\mathbb{R}}, \Phi_A)$ is a *proximal extension* of (Ω, Ξ) if for every $\xi \in \Omega$ and (ξ, φ_1), $(\xi, \varphi_2) \in K_{\mathbb{R}}$, one has $\inf_{t\in\mathbb{R}} \mathrm{d}\,(\varphi(t, \xi, \varphi_1), \varphi(t, \xi, \varphi_2)) = 0$.

If $A(\xi) \equiv A$ is a constant matrix, the systems (1.1) cannot be recurrent-proximal because recurrence requires eigenvalues of A to be purely imaginary and in this case the flow preserves distance which contradicts the proximal character. By applying the Floquet theory we conclude that if the flow (Ω, Ξ) is periodic, the family of systems (1.1) cannot be recurrent and proximal.

In [2] techniques of topological dynamics are used by Ellis and Johnson to prove the existence of recurrent-proximal systems. In [8] Johnson proved that there exist two-dimensional, almost periodic linear systems with arbitrary basis of frequencies, which are recurrent-proximal and the flow uniquely ergodic. In fact he showed, by using ideas similar to those of Glasner and Weiss in [5] where minimal skew products are constructed, that this is a generic property on a class of systems. If we consider C^k flows, some restrictions on the basis of frequencies should be imposed: using a technique based on developing a smooth version of a result of the above-mentioned paper by Glasner and Weiss [5], Nerurkar ([10], [11]) showed that in n-dimensional systems, when the C^k flow (Ω, Ξ) admits "fast periodic approximations", recurrent-proximal behaviour is generic in the C^k topology.

In this paper we study the closure of the set of systems (1.1) with null Lyapunov exponent which can be triangularized by a strong Perron transformation. First we show, under some restrictions on the flow on the basis Ω (to avoid for instance periodicity), that this class of systems coincides with the one considered by Johnson in [8] in the almost periodic case. Then we prove not only that the recurrent-proximal behaviour and uniquely ergodic character of the flow $(K_{\mathbb{R}}, \Phi_A)$ are generic properties in this set of systems, but also the singular dynamics. In fact we show that the uniquely ergodic character which is generic is the one with a singular ergodic measure, with respect to m_1, concentrated in an ergodic sheet. Then we explain the behaviour of the trajectories with respect to this ergodic sheet from which the proximal character can be also deduced.

Our techniques are based on the ergodic structure and ergodic classification of bidimensional linear systems developed in Alonso-Obaya [1] and Novo-Obaya [12]. The introduction, due to Schwartzman [14], of a certain homomorphism A_{m_0} from the group of all homotopy classes of continuous maps $\Omega \to P^1(\mathbb{R})$ to the additive reals, is of great importance for the gap-labelling theory (see Johnson [9]), and it is also helpful in our approach.

Notation and preliminaries are stated in Section 2 to finally formulate and prove the results in Section 3.

2. Preliminaries

Let m_0 be the unique ergodic measure on Ω and $\mathcal{A}_0$ the completion of the σ-algebra of the Borel sets with respect to m_0. The symbol $m_1 = m_0 \otimes r_1$ stands for the complete product measure on the corresponding σ-algebra $\mathcal{A}_1$ of $\Omega \times P^1(\mathbb{R})$. For the reader's convenience, we will recall briefly the description of the ergodic measures on $K_{\mathbb{R}}$ which project into m_0, obtained in [12]. We will refer to the absolutely continuous case when there exists an invariant measure on $K_{\mathbb{R}}$ which is absolutely continuous with respect to m_1, and the singular one when every invariant measure under the flow is singular with respect to m_1. Let us start with some definitions.

Let μ be an invariant measure under the flow Φ_A, absolutely continuous with respect to m_1, with $d\mu = p\, dm_1$. We say that μ is a *linear invariant measure* if there exist measurable functions C_1, C_2, $C_3 : \Omega \to \mathbb{R}$ and an invariant subset Ω_0 with $m_0(\Omega_0) = 1$ such that the density function $p(\xi, \varphi)$ satisfies

$$p(\xi, \varphi) = (C_1(\xi)\cos^2\varphi + C_2(\xi)\sin^2\varphi + 2\,C_3(\xi)\sin\varphi\cos\varphi)^{-1}$$

for every $(\xi, \varphi) \in \Omega_0 \times P^1(\mathbb{R})$.

The following invariant subsets are important in what follows.

Let M be an invariant subset of $K_{\mathbb{C}}$, $\pi : M \to \Omega$ the projection on the base. We say that M is a *measurable k-sheet* with respect to m_0 if there is an invariant subset $\Omega_0 \subset \Omega$ with $m_0(\Omega_0) = 1$ such that

(i) $\operatorname{card}(\pi^{-1}(\xi)) = k$ for every $\xi \in \Omega_0$.

(ii) The multivalued function defined in Ω_0 that takes $\xi \in \Omega_0$ to $\pi^{-1}(\xi) = \{(\xi, Z_1(\xi)), (\xi, Z_2(\xi)), \ldots, (\xi, Z_k(\xi))\}$ is $\mathcal{A}_0$-measurable.

When a measurable k-sheet M exists, the flow $(K_{\mathbb{C}}, \Phi_A)$ posseses a purely discontinuous invariant measure μ projecting into m_0 and satisfying $\mu(M) = 1$, defined by

$$\int_{K_{\mathbb{C}}} g\, d\mu = \frac{1}{k}\sum_{j=1}^{k}\int_{\Omega} g(\xi, Z_j(\xi))\, dm_0$$

for every $g \in C(K_{\mathbb{C}})$.

We say that M is an *ergodic k-sheet* with respect to m_0 if μ is the unique invariant Borel measure on $K_{\mathbb{C}}$ projecting into m_0 which is concentrated in M. If M is a compact subset of $K_{\mathbb{C}}$ and $\operatorname{card}(\pi^{-1}(\xi)) = k$ for every $\xi \in \Omega$, we say that M is a *closed k-sheet*. For $k = 1$ we simply refer to M as a *measurable*, *ergodic* or *closed sheet*.

The linear invariant measures are directly related to the ergodic sheets, see [1] for details.

It is shown in [12] that in the absolutely continuous case a linear invariant measure always exists. From that and a previous result of Furstenberg [4] it was deduced that if μ is an absolutely continuous measure with respect to m_1, two different options are possible:

(a.1) μ is ergodic which implies that it is necessarily a linear invariant measure, and the only invariant measure projecting into m_0.

(a.2) μ is not ergodic and every ergodic measure on $K_{\mathbb{R}}$ which projects into m_0 is concentrated in an ergodic k-sheet.

In the singular case three different options are possible:

(b.1) $(K_{\mathbb{R}}, \Phi_A)$ admits a unique ergodic measure concentrated in an ergodic sheet.

(b.2) $(K_{\mathbb{R}}, \Phi_A)$ admits two different ergodic measures, each of them concentrated in an ergodic sheet.

(b.3) $(K_{\mathbb{R}}, \Phi_A)$ admits a unique ergodic measure concentrated in an ergodic 2-sheet.

A useful technique consists in introducing a new projective bundle whose ergodic measures are concentrated in ergodic sheets. We take the new base $\widetilde{\Omega} = K_{\mathbb{R}}$ and $\widetilde{\Xi} = \Phi_A|_{\widetilde{\Omega}}$; the flow $(\widetilde{\Omega}, \widetilde{\Xi})$ extends (Ω, Ξ) and allows one to lift the equations (1.1) to $\widetilde{\Omega} \times \mathbb{C}^2$. The new coefficients are defined by the relations $\tilde{a}(\xi, X) = a(\xi)$, $\tilde{b}(\xi, X) = b(\xi)$, $\tilde{c}(\xi, X) = c(\xi)$, for every $(\xi, X) \in K_{\mathbb{R}}$. The solutions of the corresponding Riccati equations (1.2) define the new flow $\widetilde{\Phi}_A$ on the projective bundles $\widetilde{K}_{\mathbb{R}} = \widetilde{\Omega} \times P^1(\mathbb{R})$ and $\widetilde{K}_{\mathbb{C}} = \widetilde{\Omega} \times P^1(\mathbb{C})$. It is obvious that

$$\widetilde{M} = \{(\xi, X, X) \,/\, (\xi, X) \in K_{\mathbb{R}}\}$$

defines a closed ergodic sheet for every ergodic measure on $\widetilde{\Omega}$.

Finally, we give a brief summary of Schwarzmann's theory on asymptotic cycles [14] that will be needed later.

Let $\mathcal{H}$ be the set of all continuous maps $\phi : \Omega \to P^1(\mathbb{R})$. One first prove that

$$\lim_{t\to\infty} \frac{1}{t} \arg \phi(\xi_t) \ ,$$

where $\arg\phi$ denotes a continuous argument (along the trajectories) of ϕ on $\mathbb{R}$, exists and is independent of ξ for almost every $\xi \in \Omega$; we denote this limit by $A_{m_0}(\phi)$. It is easy to see that $A_{m_0}(\phi_1 + \phi_2) = A_{m_0}(\phi_1) + A_{m_0}(\phi_2)$ for each ϕ_1, $\phi_2 \in \mathcal{H}$; consequently, A_{m_0} defines a homomorphism from $\mathcal{H}$ to $\mathbb{R}$. We will denote by $\mathcal{M}$ the image of the homomorphism, i.e. $\mathcal{M} = A_{m_0}(\mathcal{H})$.

Let L be the subgroup of $\mathcal{H}$ defined by all the maps $\phi \in \mathcal{H}$ which can be written in the form $\phi(\xi) = G(\xi) \mod \pi$ where $G : \Omega \to \mathbb{R}$ is continuous. It is clear that if $\phi \in L$ then $A_{m_0}(\phi) = 0$. Moreover it is known that $\mathcal{H}/L \simeq \check{H}^1(\Omega : \mathbb{Z})$ (the group of all real Čech 1-cocycles which take integer values on integral Čech 1-cycles); hence A_{m_0} induces an additive homomorphism of $\check{H}^1(\Omega : \mathbb{Z})$ into $\mathbb{R}$ and $\mathcal{M} = A_{m_0}[\check{H}^1(\Omega : \mathbb{Z})]$ is then a countable subgroup of $\mathbb{R}$.

It is easy to prove that if the flow (Ω, Ξ) is minimal and almost periodic (in which case Ω is the hull of an almost periodic function and it may be given the structure of a compact abelian group with dense subgroup $\mathbb{R}$) $2\mathcal{M}$ coincides with the group of characters of Ω (identified with a subgroup of $\mathbb{R}$), i.e. the frequency module of the almost periodic function which generates Ω. The reason is that by a classical theorem of Bohr (see [3]) every continuous function $\phi : \Omega \to T$ is homotopic to a character (a continuous homomorphism of Ω to T).

If the flow is almost periodic but not periodic the frequency module is a dense subgroup of $\mathbb{R}$ and the same may be said of $\mathcal{M}$. Therefore this case will be included in our formulation because we will consider (Ω, Ξ) minimal and uniquely ergodic, Ω a compact metric space and $\mathcal{M}$ dense in $\mathbb{R}$.

3. Results

We will denote, as usual, $C_0(\Omega) = \{b \in C(\Omega) \,/\, \int_\Omega b(\xi)\, dm_0 = 0\}$, where m_0 is the unique normalized ergodic measure on Ω. First we consider the set of systems (1.1) with null Lyapunov exponent which can be triangularized by a strong Perron transformation. It is known that these are the systems with null Lyapunov exponent such that $\Omega \times S^1$ has a closed sheet. Let us denote by S_0 the set of matrices of such systems. Denoting by $\mathcal{P}(\Omega)$ the set of strong Perron transformations with determinant equal to one, we can write

$$\begin{aligned} S_0 = \Bigg\{ A \in C(\Omega, L(\mathbb{R}^2)) \,/\, A(\xi) = P^{-1}(\xi) \begin{pmatrix} a(\xi) & 0 \\ b(\xi) & -a(\xi) \end{pmatrix} P(\xi) \\ - P^{-1}(\xi) P'(\xi) \text{ for some } P \in \mathcal{P}(\Omega),\ a \in C_0(\Omega) \text{ and } b \in C(\Omega) \Bigg\}. \end{aligned}$$

Next we denote by S the closure of S_0 in $C(\Omega, L(\mathbb{R}^2))$ with the supremum norm. It is an easy exercise to show that in fact we get

$$\begin{aligned} S = \operatorname{cls} \Bigg\{ A \in C(\Omega, L(\mathbb{R}^2)) \,/\, A(\xi) = P^{-1}(\xi) \begin{pmatrix} 0 & 0 \\ b(\xi) & 0 \end{pmatrix} P(\xi) \\ - P^{-1}(\xi) P'(\xi) \text{ for some } P \in \mathcal{P}(\Omega) \text{ and some } b \in C(\Omega) \Bigg\}. \end{aligned}$$

From now on, recall that we assume (Ω, Ξ) minimal and uniquely ergodic and $\mathcal{M}$ (the subgroup of $\mathbb{R}$ introduced in Section 2) dense in $\mathbb{R}$.

The next theorem says that S coincides with the closure of the systems with bounded solutions.

Theorem 3.1. *Let*

$$\begin{aligned} S_0^* = \big\{ A \in C(\Omega, L(\mathbb{R}^2)) \text{ with } \operatorname{tr} A = 0 \,/ \\ \text{all the solutions of } \mathbf{z}' = A(\xi_t)\mathbf{z}, \xi \in \Omega, \text{ are bounded } \big\}. \end{aligned}$$

Then $S = \operatorname{cls}(S_0^*)$.

Proof. Let $A = \left(\begin{smallmatrix} 0 & 0 \\ b & 0 \end{smallmatrix}\right)$ with $b \in C(\Omega)$. It is clear that

$$A_\varepsilon = \begin{pmatrix} 0 & -\varepsilon b \\ b & 0 \end{pmatrix} \longrightarrow A$$

as $\varepsilon \to 0$, and all the solutions of $\mathbf{z}' = A_\varepsilon(\xi_t)\mathbf{z}$ are bounded because the change of variable

$$\mathbf{x} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\varepsilon} \end{pmatrix} \mathbf{z}$$

transforms it into

$$\mathbf{x}' = \begin{pmatrix} 0 & -\sqrt{\varepsilon}\, b(\xi_t) \\ \sqrt{\varepsilon}\, b(\xi_t) & 0 \end{pmatrix} \mathbf{x}\,.$$

Thus, we have proved $S \subset \operatorname{cls}(S_0^*)$.

In order to finish the proof we will show that $S_0^* \subset S$. Since every system with bounded solutions can be transformed by means of a strong Perron transformation into a skew-symmetric one (see [12]), we will consider them already in this form, hence let us take $B = \left(\begin{smallmatrix} 0 & -b \\ b & 0 \end{smallmatrix}\right) \in S_0^*$ with $b \in C(\Omega)$, and let λ be the average of the function b, i.e. $\lambda = \int_\Omega b\, dm_0$. We can write $b = \lambda + b_0$ with $b_0 \in C_0(\Omega)$. Since $\mathcal{M}$ is dense in $\mathbb{R}$ we can find $\beta \in \mathcal{M}$ with $|\lambda - 2\beta| < \varepsilon/2$. Then $\beta = A_{m_0}[h]$ where the representative of the class can be chosen differentiable with respect to the flow (see [14]). Therefore,

$$\beta = \lim_{t\to\infty} \frac{1}{t} \arg h(\xi_t) = \lim_{t\to\infty} \frac{1}{t} \int_0^t h'(\xi_s)\, ds = \int_\Omega h'(\xi)\, dm_0 \;,$$

which means that we can write $\beta = h_0 + h'$ with $h_0 = \beta - h' \in C_0(\Omega)$. Next we consider $g_0 = 2h_0 + b_0 \in C_0(\Omega)$. The density of the set (see [14])

$$D = \{ g \in C(\Omega) \,/\, \text{there exists } f \in C(\Omega) \text{ with } f'(\xi_t)|_{t=0} = g(\xi) \}$$

in $C_0(\Omega)$ implies that we can approximate g_0 by $\hat{g}_0$ such that the equation $R' = \hat{g}_0$ has a continuous solution along the flow and $\|g_0 - \hat{g}_0\| < \varepsilon/2$. This means that the equation $\varphi' = \hat{g}_0 + 2h'$ has a continuous solution φ along the flow on S^1, and then the systems

$$\mathbf{z}' = \begin{pmatrix} 0 & -\hat{g}_0(\xi_t) - 2h'(\xi_t) \\ \hat{g}_0(\xi_t) + 2h'(\xi_t) & 0 \end{pmatrix} \mathbf{z}$$

can be taken into $\mathbf{y}' = \mathbf{0}$ by means of the transformation

$$\mathbf{z} = \begin{pmatrix} \cos\varphi(\xi_t) & -\sin\varphi(\xi_t) \\ \sin\varphi(\xi_t) & \cos\varphi(\xi_t) \end{pmatrix} \mathbf{y} \;.$$

Finally notice that $\|b - \hat{g}_0 - 2h'\| < \varepsilon$ to conclude that $B \in S$. $\square$

Notice that included in the above proof we get that S is the set considered by Johnson in [8] (in the almost periodic case), i.e.

$$S = \operatorname{cls} \left\{ A \in C(\Omega, L(\mathbb{R}^2)) \,/\, A(\xi) = P^{-1}(\xi) \begin{pmatrix} 0 & -b(\xi) \\ b(\xi) & 0 \end{pmatrix} P(\xi) \right.$$
$$\left. - P^{-1}(\xi) P'(\xi) \text{ for some } P \in \mathcal{P}(\Omega) \text{ and some } b \in C_0(\Omega) \right\},$$

and also that S is the closure of the matrices of the systems that can be transformed, by means of a strong Perron transformation, into $\mathbf{y}' = \mathbf{0}$.

The following proposition is an easy consequence of the description of the ergodic measures obtained in [12].

Proposition 3.2. *Let $A = \left(\begin{smallmatrix} 0 & 0 \\ b & 0 \end{smallmatrix}\right)$ with $b \in C(\Omega)$ and let $(K_\mathbb{R}, \Phi_A)$ be the real projective flow induced by A. Then two options are possible:*

(i) *$(K_\mathbb{R}, \Phi_A)$ is uniquely ergodic with a singular invariant measure concentrated in an ergodic sheet.*

(ii) *$(K_\mathbb{R}, \Phi_A)$ admits infinite linear invariant mesures and $K_\mathbb{R}$ decomposes into ergodic sheets.*

Proof. First of all remark that $M = \{(\xi, \infty) \,/\, \xi \in \Omega\}$ is always an ergodic sheet of the flow and hence if $(K_\mathbb{R}, \Phi_A)$ is uniquely ergodic, the unique invariant measure is the singular one concentrated in M. If $(K_\mathbb{R}, \Phi_A)$ is not uniquely ergodic, let μ be another invariant measure. It is shown in [12] that the existence of a continuous invariant measure and an ergodic sheet implies that (ii) is satisfied. Hence we have only to examine what happens when μ is a singular measure. In this case, from the characterization of singular invariant measures, μ must be concentrated in an ergodic sheet (not passing through infinity). Let $N = \{(\xi, X(\xi)) \,/\, \xi \in \Omega\}$ be this ergodic sheet. Since the flow is translation invariant, for each $\alpha \in \mathbb{R}$ the set $N_\alpha = \{(\xi, X(\xi) + \alpha) \,/\, \xi \in \Omega\}$ is another ergodic sheet and from the existence of infinite ergodic sheets (three different in fact are enough, as proved in [1]) we conclude (ii). Finally notice that the Riccati equations in this case are $R' = b$ which implies that (i) is verified when no real measurable solution along the flow of these equations exists. $\square$

For $A = \left(\begin{smallmatrix} 0 & 0 \\ b & 0 \end{smallmatrix}\right) \in S$ with $b \in C(\Omega)$ the uniquely ergodic character is independent of the real, complex or extended flow considered.

Proposition 3.3. *Let $A = \left(\begin{smallmatrix} 0 & 0 \\ b & 0 \end{smallmatrix}\right)$ with $b \in C(\Omega)$. The following statements are equivalent:*

(i) $(K_\mathbb{C}, \Phi_A)$ *is uniquely ergodic.*
(ii) $(K_\mathbb{R}, \Phi_A)$ *is uniquely ergodic.*
(iii) $(\widetilde{K}_\mathbb{C}, \widetilde{\Phi}_A)$ *is uniquely ergodic.*

Proof. (i) $\Rightarrow$ (ii) is obvious. (iii) $\Rightarrow$ (ii) follows from the fact that $K_\mathbb{R}$ is the base of $\widetilde{K}_\mathbb{C}$.

Let now assume that $(K_\mathbb{R}, \Phi_A)$ is uniquely ergodic. It is proved in [12] that the existence of an invariant measure for $(K_\mathbb{C}, \Phi_A)$ with $\mu(K_\mathbb{R}) = 0$ implies that $(K_\mathbb{R}, \Phi_A)$ admits a linear invariant measure, which means that in the singular case every ergodic measure is concentrated on the real projective bundle. From this fact and Proposition 3.2 we conclude that $(K_\mathbb{C}, \Phi_A)$ is uniquely ergodic and (ii) $\Rightarrow$ (i) is proved.

To show (ii) $\Rightarrow$ (iii) first notice that if $(K_\mathbb{R}, \Phi_A)$ is uniquely ergodic then the extended flow $(\widetilde{K}_\mathbb{R}, \widetilde{\Phi}_A)$ is also uniquely ergodic because, as proved in [12], the existence of a linear invariant measure for $(\widetilde{K}_\mathbb{R}, \widetilde{\Phi}_A)$ would imply the existence of a linear invariant measure for $(K_\mathbb{R}, \Phi_A)$, and Proposition 3.2 is true changing the flow by the extended one. Now the same arguments used to show that (ii) $\Rightarrow$ (i) prove that if $(\widetilde{K}_\mathbb{R}, \widetilde{\Phi}_A)$ is uniquely ergodic then $(\widetilde{K}_\mathbb{C}, \widetilde{\Phi}_A)$ is uniquely ergodic (we have only to change the base Ω to $K_\mathbb{R}$). $\square$

The next theorem says that not only the recurrent behaviour and uniquely ergodic character but also the singular dynamics is a generic property in S.

Theorem 3.4. *There is a residual set R of S such that if $A \in R$, then $(K_\mathbb{R}, \Phi_A)$ is minimal and uniquely ergodic with a singular measure concentrated in an ergodic sheet.*

Proof. First we will prove that there is a residual set R_1 of S such that if $A \in R_1$, then $(K_\mathbb{R}, \Phi_A)$ is minimal. It can be shown that $\mathbf{z}' = \left(\begin{smallmatrix} 0 & -b \\ b & 0 \end{smallmatrix}\right)\mathbf{z}$ is recurrent if $k\alpha \notin \mathcal{M}$ for every $k \in \mathbb{N}$, when $\alpha = \int_\Omega b(\xi)\, dm_0 \neq 0$. Since $\mathcal{M}$ is a dense countable subgroup of $\mathbb{R}$ we deduce from Theorem 3.1 that the subset of S where the flow is minimal is dense in S. The rest of the construction of R_1 is detailed in Proposition 3.6 of [8].

Now we consider the sets

$$S_1 = \{A \in S \,/\, (K_\mathbb{C}, \Phi_A) \text{ is u.e.}\}\,,$$

$$S_2 = \{A \in S \,/\, (K_\mathbb{R}, \Phi_A) \text{ is u.e.}\}\,,$$

$$S_3 = \{A \in S \,/\, (\widetilde{K}_\mathbb{C}, \widetilde{\Phi}_A) \text{ is u.e.}\}\,.$$

From Propositions 3.2 and 3.3 and since the set

$$F = \{b \in C(\Omega) \,/\, R' = b(\xi_t) \text{ has no measurable solution along the flow}\}$$

is dense in $C(\Omega)$ (notice that every $b \in C(\Omega)$ with positive average is in F) we deduce that S_1, S_2, S_3 are dense subsets of S. The proof that those are residual sets is standard [5] and we will give only a sketch of it. Let μ be an invariant measure under the flow $(K_{\mathbb{C}}, \Phi_A)$, and denote by $C_0(K_{\mathbb{C}}) = \{f \in C(K_{\mathbb{C}}) \,/\, \int_{K_{\mathbb{C}}} f\, d\mu = 0\}$. Recall that $(K_{\mathbb{C}}, \Phi_A)$ is uniquely ergodic if and only if for every $f \in C_0(K_{\mathbb{C}})$ and $(\xi, Z) \in K_{\mathbb{C}}$ the limit $(1/t)\int_0^t f(\Phi_A(s,\xi,Z))\, ds \to 0$ as $t \to \infty$. Let $\{f_j\}_{j\in\mathbb{N}}$ be a countable dense subset of $C_0(K_{\mathbb{C}})$; for each $j, n \in \mathbb{N}$ we consider the set $V_{j,1/n} = \left\{A \in S \,/\, \exists\, \hat{t} \neq 0 \text{ with } \left|(1/\hat{t})\int_0^{\hat{t}} f_j(\Phi_A(s,\xi,Z))\, ds\right| < 1/n \quad \forall\, (\xi, Z) \in K_{\mathbb{C}}\right\}$. It is not hard to see that $V_{j,1/n}$ is open and $S_1 = \cap_{j=1}^{\infty} \cap_{n=1}^{\infty} V_{j,1/n}$, from which we deduce that S_1 is a residual set of S. The proof is completely analogous for S_2 and S_3. Now we take $R_2 = S_1 \cap S_2 \cap S_3$, which is also a residual set of S: if $A \in R_2$ then $(K_{\mathbb{R}}, \Phi_A)$ is uniquely ergodic. Let μ be the unique invariant measure; according to the caracterization of the invariant measures recalled in Section 2 we have three possibilities:

(i) μ is absolutely continuous and then linear.
(ii) μ is singular and is concentrated in an ergodic sheet.
(iii) μ is singular and is concentrated in an ergodic 2-sheet.

(i) contradicts the fact that if $A \in R_2$ then $(K_{\mathbb{C}}, \Phi_A)$ is uniquely ergodic, because a linear measure provides two complex ergodic sheets and therefore two invariant singular measures. The existence of a 2-sheet of $(K_{\mathbb{R}}, \Phi_A)$ implies the existence of two different ergodic sheets on $(\widetilde{K}_{\mathbb{C}}, \widetilde{\Phi}_A)$ which contradicts the fact that this extended flow is uniquely ergodic for $A \in R_2$; thus (iii) cannot happen. Consequently, if $A \in R_2$ the unique invariant measure is singular and is concentrated in an ergodic sheet. Finally, by considering the residual set $R = R_1 \cap R_2$ we deduce the result. $\square$

From now on we will assume that we are in the singular case with an ergodic measure concentrated in an ergodic sheet. The following theorem explains the oscillations of the trajectories from one to the other side of the ergodic sheet.

Let φ_1, $\varphi_2 \in P^1(\mathbb{R})$, and denote by (φ_1, φ_2) the open arc obtained by moving clockwise on $P^1(\mathbb{R})$, and by $d(\varphi_1, \varphi_2)$ its length, i.e.

$$d(\varphi_1, \varphi_2) = \begin{cases} \varphi_2 - \varphi_1 & \text{if } 0 \le \varphi_1 \le \varphi_2 < \pi, \\ \pi + \varphi_2 - \varphi_1 & \text{if } 0 \le \varphi_2 < \varphi_1 < \pi\,. \end{cases}$$

Theorem 3.5. *Let us assume that $(K_{\mathbb{R}}, \Phi_A)$ is uniquely ergodic with a singular measure concentrated in an ergodic sheet $M = \{(\xi, \phi(\xi)) \,/\, \xi \in \Omega\}$. There exists an invariant subset $\Omega_0 \subset \Omega$ with $m_0(\Omega_0) = 1$ such that for each $\xi \in \Omega_0$ we can find two sequences of real numbers $(t_n^1)_{n\in\mathbb{N}}$, $(t_n^2)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} |t_n^i| = \infty$, $i = 1, 2$, satisfying $d(\phi(\xi_{t_n^1}), \varphi(t_n^1, \xi, \varphi)) \to 0$ as $n \to \infty$ and $d(\varphi(t_n^2, \xi, \varphi), \phi(\xi_{t_n^2})) \to 0$ as $n \to \infty$ for every $\varphi \in P^1(\mathbb{R})$.*

Proof. First of all we define $D_1(\xi, \varphi) = \inf_{t\in\mathbb{R}}\{d(\phi(\xi_t), \varphi(t, \xi, \varphi))\}$ for $\xi \in \Omega$ and $\varphi \in P^1(\mathbb{R})$. It is not hard to show that D_1 is a measurable invariant function (by continuity of the solutions of the equation and of the flow Ξ this infimum coincides with the one taking t within the countable set $\mathbb{Q}$), $D_1(\xi, \phi(\xi)) = 0$ and $D_1(\xi, \varphi_1) \le D_1(\xi, \varphi_2)$ if $\varphi_1 \in (\phi(\xi), \varphi_2)$. We will show now that there is a subset

$\Omega_0 \subset \Omega$ with $m_0(\Omega_0) = 1$ such that if $\xi \in \Omega_0$ then $D_1(\xi, \varphi) = 0$ for every $\varphi \in P^1(\mathbb{R})$. First let us assume that D_1 takes at least two different values on $P^1(\mathbb{R}) - \{\phi(\xi)\}$. For each rational q we denote by $\Omega(q)$ the set of points $\xi \in \Omega$ for which there exist $\varphi_\pm(\xi) \in P^1(\mathbb{R}) - \{\phi(\xi)\}$ with $D_1(\xi, \varphi_-(\xi)) < q < D_1(\xi, \varphi_+(\xi))$. It is obvious that $\Omega(q)$ is invariant and $m_0(\cup_{q\in\mathbb{Q}}\Omega(q)) = 1$, hence there is $q_0 \in \mathbb{Q}$ with $m_0(\Omega(q_0)) = 1$. The set $C = \{(\xi, \varphi) \in K_\mathbb{R} \,/\, \xi \in \Omega \text{ and } D_1(\xi, \varphi) < q_0\}$ is measurable and invariant; moreover if $(\xi, \varphi) \in C$ one has that $(\xi, \varphi') \in C$ for every $\varphi' \in (\phi(\xi), \varphi)$. For $\xi \in \Omega(q_0)$ we will denote by $\phi_0(\xi)$ the unique element of $P^1(\mathbb{R})$ satisfying

$$\begin{aligned} D_1(\xi, \varphi) &\le q_0 \quad \text{for every } \varphi \in [\phi(\xi), \phi_0(\xi)) \\ D_1(\xi, \varphi) &> q_0 \quad \text{for every } \varphi \in (\phi_0(\xi), \phi(\xi)]\,. \end{aligned}$$

It follows from Fubini's theorem that the map

$$l\,:\,\Omega \to \mathbb{R}\,, \quad \xi \to \int_{P^1(\mathbb{R})} \chi_C(\xi, \varphi)\, dr_1(\varphi)$$

is measurable and we get that $\phi_0(\xi) = \phi(\xi) + l(\xi)$ on $P^1(\mathbb{R})$. Therefore, the set $M_0 = \{(\xi, \phi_0(\xi)) \,/\, \xi \in \Omega\}$ defines a second ergodic sheet of $K_\mathbb{R}$, but this contradicts our hypothesis. Hence we conclude that there exists an invariant subset of complete measure Ω_0 and a real number α such that $D_1(\xi, \varphi) = \alpha$ for every $\xi \in \Omega_0$ and $\varphi \in P^1(\mathbb{R}) - \{\phi(\xi)\}$; to see that $\alpha = 0$ notice that $D_1(\xi, \varphi) \le d(\phi(\xi), \varphi)$.

Now for each $\xi \in \Omega_0$ we select a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of elements from $P^1(\mathbb{R}) - \{\phi(\xi)\}$ with $\varphi_{n+1} \in (\varphi_n, \phi(\xi))$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \varphi_n = \phi(\xi)$. Since $D_1(\xi, \varphi) = 0$, for each n we can find $t_n^1 \in \mathbb{R}$ such that $d(\phi(\xi_{t_n^1}), \varphi(t_n^1, \xi, \varphi)) \le 1/n$ for every $\varphi \in (\phi(\xi), \varphi_n)$. This means that $\lim_{n\to\infty} d(\phi(\xi_{t_n^1}), \varphi(t_n^1, \xi, \varphi)) = 0$ for every $\varphi \in P^1(\mathbb{R})$. The existence of the other sequence $(t_n^2)_{n\in\mathbb{N}}$ is obtained arguing in a similar way with the map $D_2(\xi, \varphi) = \inf_{t\in\mathbb{R}}\{d(\varphi(t, \xi, \varphi), \phi(\xi_t))\}$. □

Thus, depending on the sign of the time sequences obtained in the above theorem, two different situations are possible. If they have opposite signs at infinity, then, as time goes from $-\infty$ to $+\infty$ the trajectories move from one side of the ergodic sheet to the other; whereas, if they have the same sign at infinity, the trajectories oscillate between these sides.

Since distal pairs move along the trajectories and are preserved on the limits, it is easy to show that the existence of a distal pair (ξ_0, φ_1), (ξ_0, φ_2) implies the existence of a distal pair $(\xi, \varphi_1(\xi))$, $(\xi, \varphi_2(\xi))$ for every $\xi \in \Omega$. From this we conclude that the singular dynamics obtained in the above theorem precludes the existence of a distal pair, and in consequence that in the residual set R_2 obtained in Theorem 3.4 the flow $(K_\mathbb{R}, \Phi_A)$ is a proximal extension of (Ω, Ξ). In [8] Johnson deduces the proximal behaviour from the minimal character of the flow and the unboundedness of the solutions.

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