RINGS OF WEAK DIMENSION ONE AND SYZYGETIC IDEALS

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Abstract. We prove that rings of weak dimension one are the rings with all (three-generated) ideals syzygetic. This leads to a characterization of these rings in terms of the André-Quillen homology.

Let $I$ be an ideal of a commutative ring $A$. There is a canonical morphism of graded $A$-algebras $\alpha : S(I) \to R(I)$ from the symmetric algebra of $I$ onto its Rees algebra. The ideal $I$ is said to be of linear type if $\alpha$ is an isomorphism. If $\alpha_2 : S_2(I) \to I^2$ is an isomorphism, $I$ is said to be syzygetic.

In [C] (Theorem 4), Costa showed that a domain $A$ is Prüfer if and only if $I$ is of linear type for every two-generated ideal $I$ of $A$ and $I$ is syzygetic for every three-generated ideal $I$ of $A$. In this note we show that the preliminary hypothesis that $A$ is a domain can be removed by changing the Prüfer condition to the condition $wd(A) \leq 1$, weak dimension of $A$ one or less. Moreover, the condition that every two-generated ideal of $A$ be of linear type is not necessary. Concretely,

Theorem 1. Let $A$ be a commutative ring. The following conditions are equivalent:

i) $wd(A) \leq 1$.

ii) Every ideal of $A$ is of linear type.

iii) Every ideal of $A$ is syzygetic.

iv) Every three-generated ideal of $A$ is syzygetic.

Recall that $wd(A)$ is the supremum of the flat dimensions of all $A$-modules. The von Neumann regular rings are those of weak dimension zero. Semidirect rings (i.e. rings with all their finitely generated ideals projective) have weak dimension one or less. In fact, $A$ is a semidirect ring if and only if $wd(A) \leq 1$ and $A$ is coherent. A semidirect domain is called a Prüfer ring. For a domain $A$, to be Prüfer is equivalent to $wd(A) \leq 1$ (see [B] or [R]). In particular, if $A$ is a domain, Theorem 1 characterizes Prüfer rings as domains with every three-generated ideal being syzygetic.

Before proving Theorem 1 we need the following two lemmas.

Lemma 2. Let $(A, m)$ be a local ring. Suppose that there exist two nonzero elements $a, b \in A$ with $ab = 0$. If $(a)$ is syzygetic, then $(a, b)$ is not.

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Proof. Let \( I = (a, b) \subset \mathfrak{m} \) be the ideal of \( A \) generated by the zero divisors \( a, b \). If \( x \in (a) \cap (b) \), \( x = ac = bd \), and multiplying by \( a, c \in (0 : a^2) \). Since \( (a) \) is syzygetic, \( c \in (0 : a) \) (see [V], page 31) and \( x = ca = 0 \). Therefore, \( (a) \cap (b) = 0 \).

Let \( 0 \to Z_1 \to A^2 \overset{f}{\to} I \to 0 \) be the free presentation of \( I \) defined by \( f((1, 0)) = a, f((0, 1)) = b \). Consider \( 0 \to N \to A[X, Y] \overset{s}{\to} \mathbf{R}(I) \to 0 \), the induced free presentation of \( \mathbf{R}(I) = \oplus_{q \geq 0} I^q t^q \), defined by \( s(X) = at, s(Y) = bt \).

If \( I = (a, b) \) were syzygetic, the quadratic relation \( XY \) on \( a, b \) could be written in terms of linear relations on \( a, b \) ([V], page 29), i.e.

\[
XY = (a_1X + b_1Y)(c_1X + d_1Y) + \cdots + (a_rX + b_rY)(c_rX + d_rY)
\]

with \( c_iX + d_iY \in N_1 = Z_1 \). In particular, \( c_1a = -d_1b \in (a) \cap (b) = 0 \). Therefore, \( c_1a = d_1b = 0 \) and if \( a, b \neq 0 \), then \( c_i, d_i \) would be zero divisors, in particular, elements of \( \mathfrak{m} \). Comparing the coefficients of \( XY \) on both sides of (1) we would get the contradiction \( 1 = \sum c_i a_i d_i + b_i c_i \in \mathfrak{m} \).

Lemma 3. Let \((A, \mathfrak{m}, k)\) be a local ring. Let \( I \) be a nonprincipal finitely generated ideal of \( A \). If \( I \) is syzygetic, then \( I^2 \) is not.

Proof. As \( I \) is not principal, \( \dim_k(I/\mathfrak{m}I) = n > 1 \). By hypothesis, \( \alpha_2 \) is an isomorphism and hence, \( \alpha_2 \otimes 1_k \) is also an isomorphism. Therefore,

\[
\dim_k(I^2/\mathfrak{m}I^2) = \dim_k(S^2_k(I/\mathfrak{m}I)) = \frac{n(n + 1)}{2} = p.
\]

Thus, \( \dim_k(S^2_k(I^2/\mathfrak{m}I^2)) = \frac{p(p + 1)}{2} = \frac{n^2 + n + 2}{8} \). Since \( \alpha_4 \) is an epimorphism, \( \alpha_4 \otimes 1_k \) is also an epimorphism. So, one has

\[
\dim_k(I^4/\mathfrak{m}I^4) \leq \dim_k(S^4_k(I/\mathfrak{m}I)) = \frac{n(n + 1)(n^2 + 5n + 6)}{24}.
\]

Finally, one observes that if \( n \neq 0, 1 \), then \( 3(n^2 + n + 2) > (n^2 + 5n + 6) \). In particular, \( S^2_k(I^2/\mathfrak{m}I^2) \not\cong I^4/\mathfrak{m}I^4 \) and \( S^4_k(I^2) \not\cong I^4 \).

Proof of the theorem. Recall \( wd(A) \leq 1 \) is equivalent to every ideal of \( A \) being flat ([R]). If \( I \) is a flat ideal of \( A \), then \( I \) is an ideal of linear type (see Proposition 3 [MR] or [P]). This proves \( i) \Rightarrow ii) \). Let us show that \( iv) \Rightarrow i) \).

Let \( I \) be an ideal of \( A \). To show \( I \) is flat, one can suppose \( I \) is finitely generated, since any ideal is the direct limit of finitely generated ideals, and the direct limit of flat modules is again a flat module. Write \( I = (x_1, \ldots, x_n) \) and let us see that \( I_\mathfrak{m} \) is a free \( A_\mathfrak{m} \)-module for every maximal ideal \( \mathfrak{m} \) of \( A \). If \( I \nsubseteq \mathfrak{m} \), \( I_\mathfrak{m} = A_\mathfrak{m} \). If \( I \subseteq \mathfrak{m} \), consider \( J = (x_1, x_2) \subseteq I \). By hypothesis \( iv) \), \( J \) and \( J^2 \) are syzygetic ideals of \( A \). Localizing at \( \mathfrak{m} \), Lemma 3 provides an element \( z \in J \) with \( J_\mathfrak{m} = (\frac{z}{1}) \). Therefore \( I_{\mathfrak{m}} = (1, \frac{x_1}{1}, \ldots, \frac{x_n}{1}) \). Now, take \( J' = (z, x_3) \subseteq I \). Repeating the process we deduce that \( I_{\mathfrak{m}} \) is a principal ideal of \( A_\mathfrak{m} \). By hypothesis \( iv) \) and Lemma 2, \( A_\mathfrak{m} \) is a domain, in particular, \( I_{\mathfrak{m}} \) is an ideal generated by a nonzero divisor, i.e. a free \( A_{\mathfrak{m}} \)-module.

Remark 4. From Lemma 2 and Costa’s Theorem 3 of [C] one deduces that for a commutative ring \( A \) to be a locally an integrally closed domain is equivalent to being of linear type for every two-generated ideal of \( A \). Moreover, the same example given by Costa in his paper shows that this last condition is strictly stronger than every two-generated ideal of \( A \) being syzygetic.
In terms of the André-Quillen homology (see [A]) and as a corollary of Theorem 1, rings of weak dimension one are characterized as follows:

**Corollary 5.** Let $A$ be a commutative ring. The following conditions are equivalent:

i) $wd(A) \leq 1$.

ii) $H_2(A, B, \cdot) = 0$ for every quotient ring $B = A/I$ of $A$.

iii) $H_2(A, B, B) = 0$ for every quotient ring $B = A/I$ of $A$ by a three-generated ideal $I$ of $A$.

**Proof.** It follows from the fact that if $I$ is an ideal of $A$, $B = A/I$ and $\alpha_2 : S_2(I) \to I^2$ is the canonical morphism, then $H_2(A, B, B) = \text{Ker} \alpha_2$. Moreover, if $I$ is syzygetic, $H_2(A, B, W) = \text{Tor}_1^B(I/I^2, W)$ for any $B$-module $W$ (see, for instance, [BR]).

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**References**


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