

BOUNDARIES OF ROTATION SETS FOR HOMEOMORPHISMS OF THE n -TORUS

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ABSTRACT. We construct a C^ω diffeomorphism of the 3-torus whose rotation set is not closed. We prove that the rotation set of a homeomorphism of the n -torus contains the extreme points of its closed convex hull. Finally, we show that each pseudo-rotation set is closed for torus homeomorphisms.

1. INTRODUCTION

H. Poincaré introduced the idea of the rotation set of an orientation preserving circle homeomorphism and proved that it consists of a single real number ([P]). In 1979 S. Newhouse, J. Palis, and F. Takens introduced the idea of a rotation set for maps of the circle (homotopic to the identity) and proved that this set is always a closed interval ([N-P-T]). M. Handel has shown that the analogous rotation set for an annulus homeomorphism (isotopic to the identity) must be closed ([H]). It follows from work of J. Franks (but see also [M-Z]) that the rotation set of a 2-torus homeomorphism must contain the extreme points and the (2-dimensional) interior of its closed convex hull ([F]). (We extend this extreme point containment result to arbitrary n -tori in Theorem 4.8, using a functional analytic approach.)

Whether or not the rotation set of a 2-torus homeomorphism must be closed remains an open question. But M. Misiurewicz and K. Ziemian have shown that vectors on the boundary of the rotation set of a homeomorphism $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ need not be the expected value of the displacement by f for any ergodic invariant probability measure ([M-Z]). We caution the reader that the definition of rotation set used here corresponds to the “point rotation set” in [M-Z].

Our main result is the construction of a C^ω diffeomorphism $F : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ isotopic to the identity with a lift $\tilde{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with rotation set $\rho(\tilde{F})$ which is not closed (Theorem 3.2). (Note: throughout this article lifts of a map G to the universal cover will be written \tilde{G} .) The rotation set of \tilde{F} is found explicitly which is unusual for non-trivial analytic examples of rotation behavior.

Next we analyze $\rho(\tilde{F})$ by proving two theorems about the general nature of boundaries of rotation sets for homeomorphisms of $\tilde{G} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ($n \geq 1$) which are lifts of toral maps. First we use the Krein-Milman theorem to show that $\rho(\tilde{G})$ contains the extreme points of its closed convex hull (Theorem 4.8). And second

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we show that the “pseudo-rotation set” (defined rigorously below) $\rho_\psi(\tilde{G})$ (which contains $\rho(\tilde{G})$) must be closed (Theorem 5.1). Roughly speaking, the latter set measures the rotation set that a computer might calculate. Such pseudo-rotation vectors are easy to compute and behave well on chain transitive sets, where they constitute a closed convex set. See [B-S] for more on pseudo rotation sets for annulus homeomorphisms. Lastly, we exhibit the pseudo rotation set, $\rho_\psi(\tilde{F})$, for the example constructed above. Our work with rotation sets of 3-torus homeomorphisms has led us to the following conjecture:

Conjecture 1.1. *The rotation set of a lift of a 3-torus homeomorphism, isotopic to the identity, contains the closure of its (3-dimensional) interior.*

2. DEFINITIONS

For general definitions consult an introductory text on dynamical systems such as [G-H]. We will take as our model of the n -torus the space $\mathbb{T}^n = \overbrace{S^1 \times \cdots \times S^1}^n$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. As universal cover we will use the map $\Pi : \mathbb{R}^n \rightarrow \mathbb{T}^n$ given by $\Pi(x_1, \dots, x_n) = (\exp 2\pi i x_1, \dots, \exp 2\pi i x_n)$.

Now assume that $G : \mathbb{T}^n \rightarrow \mathbb{T}^n$ is isotopic to the identity and $\tilde{G} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a (continuous) lift of G . Then $\rho(\tilde{G}, x)$ denotes the *rotation set of $x \in \mathbb{R}^n$ under \tilde{G}* , and is given by

$$\rho(\tilde{G}, x) = \text{LIM} \left\{ \frac{\tilde{G}^n(x) - x}{n} \mid n \in \mathbb{N} \right\}.$$

Here we use “LIM” to denote “the set of limit points”. The *rotation set of \tilde{G}* is the union

$$\rho(\tilde{G}) = \bigcup_{x \in \mathbb{R}^n} \rho(\tilde{G}, x).$$

For any map $f : X \rightarrow X$ on a compact metric space, let $\mathcal{M}_I(f)$ denote the set of f -invariant Borel probability measures, which forms a compact convex subset in the topological vector space of signed (regular Borel) measures on X . If $G : \mathbb{T}^n \rightarrow \mathbb{T}^n$, with a lift \tilde{G} , define $\mathcal{M}_I(\tilde{G})$ to be the set of lifts of regular Borel probability measures on \mathbb{T}^n . A Borel measure $\tilde{\mu}$ on \mathbb{R}^n is the *lift* of the probability measure μ on \mathbb{T}^n if for each fundamental domain Q and Borel set $E \subset \mathbb{R}^n$, $\tilde{\mu}(E \cap Q) = \mu(\Pi(E \cap Q))$. For example, the Haar measure on \mathbb{T}^2 lifts to Lebesgue measure on \mathbb{R}^2 .

Define the *mean rotation number* by

$$\rho_{\tilde{\mu}}(\tilde{G}) = \int_Q (\tilde{G} - Id) d\tilde{\mu},$$

where Q is an arbitrary fundamental domain in the covering space of \mathbb{T}^n . When μ is ergodic for G , we will speak of the *ergodic mean rotation number*. Note that if μ is a G -invariant Borel probability measure on \mathbb{T}^n , then $\tilde{\mu}$ is a \tilde{G} -invariant Borel (infinite!) measure on \mathbb{R}^n that is invariant under deck translations.

Now define the *mean rotation set*

$$\rho_{\text{mes}}(\tilde{G}) = \bigcup_{\mu \in \mathcal{M}_I(\tilde{G})} \rho_{\tilde{\mu}}(\tilde{G}).$$

Define an ϵ pseudo-orbit to be an infinite sequence $\{z_k\}_{k \geq 0} \subset \mathbb{R}^n$ such that $\|z_{k+1} - \tilde{G}(z_k)\| < \epsilon$ for $k \geq 0$. An ϵ -chain is a finite sequence z_0, z_1, \dots, z_m with $\|z_{k+1} - \tilde{G}(z_k)\| < \epsilon$ for all k with $0 \leq k < m$. Define the pseudo-rotation set of the ϵ pseudo-orbit $\{z_k\}$ by $\rho_\psi(\{z_k\}) = \text{LIM}\{(z_k - z_0)/k : k \in \mathbb{N}\}$. The union over all such pseudo-rotation sets, for a given ϵ , is the ϵ pseudo-rotation set, $\rho_\psi(\tilde{G}, \epsilon)$. Then $r \in \mathbb{R}^n$ is a pseudo-rotation vector if it lies in $\rho_\psi(\tilde{G}, \epsilon)$ for each ϵ . The analogous pseudo-rotation set for an annulus homeomorphism, in fact, equals the closure of the rotation set ([B-S]). Thus, $\rho_\psi(\tilde{G}) = \rho(\tilde{G})$, since M. Handel has shown that the true rotation set is closed ([H]). It is not hard to find examples of lifts, \tilde{G} , of homeomorphisms of \mathbb{T}^2 for which $\rho(\tilde{G}) \neq \rho_\psi(\tilde{G})$.

We need these next definitions to state our theorem concerning extreme points. Let X be a subset of a topological vector space \mathbb{E} . An extreme point $p \in X$ is any point which meets the following criterion: if $p = tx + (1 - t)y$ for points x and y in X and $0 < t < 1$, then $x = p = y$. A subset C of a topological vector space is convex if whenever $x, y \in C$ then $tx + (1 - t)y \in C$, for all $0 \leq t \leq 1$. A topological vector space \mathbb{E} is locally convex if zero has a neighborhood base of convex sets. The closed convex hull $\langle X \rangle$ of X is the intersection of all closed convex sets containing X . The closed convex hull equals the closure of the convex hull. The convex hull of X equals the set of all finite linear combinations of the form $\sum t_i x_i$, where $x_i \in X$ and $\sum t_i = 1$. If the linear space \mathbb{E} is a metric space, then $\langle X \rangle$ consists of all convergent infinite series of the form $\sum t_i x_i$ with $x_i \in X$ for $i = 1, 2, \dots$ and $\sum t_i = 1$.

3. A ROTATION SET WHICH IS NOT CLOSED

We first construct a 1-parameter family of C^ω diffeomorphisms $\{f_\lambda : 0 \leq \lambda \leq 1\}$ of \mathbb{T}^2 with lifts $\{\tilde{f}_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}^2\}$, for which the diameter of the rotation set is not continuous in the parameter. (No such family of circle maps or annulus homeomorphisms exists.) At $\lambda = \frac{1}{2}$, $\{f_\lambda | \lambda \in [0, 1/2]\}$ undergoes a double “blue sky” bifurcation. For $0 \leq \lambda < \frac{1}{2}$, f_λ has no periodic orbits. And the rotation set of the lift $\rho(\tilde{f}_\lambda) = \{(\frac{1}{4}, \tau(\lambda))\}$, where $\tau(\lambda) > 0$ and $\tau(\lambda) \rightarrow 0$ as $\lambda \rightarrow \frac{1}{2}^-$. The map $f_{\frac{1}{2}}$ has a semi-stable orbit of period two and a parallel semi-stable fixed circle, and $\rho(\tilde{f}_{\frac{1}{2}}) = \{(0, 0), (\frac{1}{2}, 0)\}$.

Each member of our family of lifts, $\{\tilde{f}_\lambda\}$, is the time-one map of a flow φ_λ^t generated by the C^ω vector field $X_\lambda = (X_\lambda^1, X_\lambda^2)$ on \mathbb{R}^2 , where

$$X_\lambda^1(x, y) = \frac{1}{4}(1 - \cos(2\pi y)) \quad \text{and} \quad X_\lambda^2(x, y) = 2 - \cos(4\pi y) - 2\lambda.$$

Clearly the vectorfields $\{X_\lambda\}$ satisfy the following:

- 3.1) $X_\lambda(x, y)$ is independent of x .
- 3.2) $X_\lambda^1(x, y) = X_\lambda^1(y)$ is independent of λ , is an even function of y , and $X_\lambda^1(y) = \frac{1}{2} - X_\lambda^1(\frac{1}{2} - y)$.
- 3.3) $X_\lambda^2(y) = X_\lambda^2(y + \frac{1}{2})$.

Lemma 3.1. For all $0 \leq \lambda < \frac{1}{2}$, $\rho(\tilde{f}_\lambda) = \{(\frac{1}{4}, \tau(\lambda))\}$, where $\tau : [0, \frac{1}{2}] \rightarrow \mathbb{R}$ is a continuous non-increasing function such that $\tau(\lambda) = 0$ if and only if $\lambda = \frac{1}{2}$. And when $\lambda = \frac{1}{2}$, $\rho(\tilde{f}_\lambda) = \{(0, 0), (\frac{1}{2}, 0)\}$.

Proof. We will need the notation

$$\rho_x(\tilde{f}_\lambda) = \text{proj}_x \rho(\tilde{f}_\lambda) \quad \text{and} \quad \rho_y(\tilde{f}_\lambda) = \text{proj}_y \rho(\tilde{f}_\lambda),$$

where $\text{proj}_x, \text{proj}_y : \mathbb{R}^2 \rightarrow \mathbb{R}$ are projections onto the x - and y -axes. $\rho(\tilde{f}_\lambda)(x, y) = \rho(\varphi_\lambda^t)(x, y)$ for all (x, y) ([FM]). Thus by 3.1 $\rho(\tilde{f}_\lambda)(x, y)$ is independent of x . Fix (x_0, y_0) and let $\varphi_\lambda^t(x_0, y_0) = (x(t), y(t))$. So $(x'(t), y'(t)) = (X_\lambda^1, X_\lambda^2)$.

First assume $0 \leq \lambda < \frac{1}{2}$. Because X_λ^2 is a *positive* function only of y , by 3.1 $\rho(\tilde{f}_\lambda)(x, y)$ is independent of (x, y) . We first show $\rho_x(\tilde{f}_\lambda) = \frac{1}{4}$ for λ in this range. It follows from 3.3 that there exists $T > 0$ (depending on y_0) such that

$$y(t + T) = y + \frac{1}{2}$$

for all t . Using 3.2) the reader should check that

$$x(t) = x_0 + \int_0^t X_\lambda^1(y(s)) ds = x_0 + \frac{1}{2}t - \int_T^{t+T} X_\lambda^1(y(s)) ds.$$

Thus

$$x(t) = x_0 + \frac{1}{2}t - x(t + T) + x(t)$$

and

$$\begin{aligned} \rho_x(\tilde{f}_\lambda) &= \text{LIM} \left\{ \frac{x(t) - x_0}{t} \mid t \geq 0 \right\} \\ &= \text{LIM} \left\{ \frac{1}{2} - \frac{x(t+T)}{t} + \frac{x(T)}{t} \mid t \geq 0 \right\} \\ &= \frac{1}{2} - \text{LIM} \left\{ \frac{x(t+T) - x_0}{t} \mid t \geq 0 \right\} \\ &= \frac{1}{2} - \rho_x(\tilde{f}_\lambda). \end{aligned}$$

So $\rho_x(\tilde{f}_\lambda) = \frac{1}{4}$ for λ in this range.

Next we consider $\rho_y(\tilde{f}_\lambda)$. The equation $y'(t) = X_\lambda^2(y)$ generates a flow on \mathbb{R} . The time one map of this flow covers a circle homeomorphism with well-defined rotation number ([P]). In fact this rotation number is $\rho_y(\tilde{f}_\lambda)$. In the statement of this lemma we call it $\tau(\lambda)$. But rotation numbers of circle homeomorphisms depend continuously on the homeomorphism. So τ is continuous. Now for each y , $X_\lambda^2(y)$ is a positive strictly decreasing function of λ in this range. However, $X_{\frac{1}{2}}^2$ has a zero at $y = 0$ (as well as many others). This means that $\rho_y(\tilde{f}_{\frac{1}{2}}) = 0$ and consequently $\tau(\lambda) \rightarrow 0$ as $\lambda \rightarrow \frac{1}{2}$.

Now fix $\lambda = \frac{1}{2}$. Notice $X_{\frac{1}{2}}^1(x, y) > 0$ and $X_{\frac{1}{2}}^2(x, y)$ has singularities only at $\{\frac{n}{2} \mid n \in \mathbb{Z}\}$. In terms of the torus coordinates $(\underline{x}, \underline{y}) = (e^{2\pi i x}, e^{2\pi i y})$ on \mathbb{T}^2 , the ω -limit set under $f_{\frac{1}{2}}$ of each point in \mathbb{T}^2 is either contained in $\{\underline{y} = 1\}$ or $\{\underline{y} = -1\}$. But, for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, $X_{\frac{1}{2}}^1(x, n) = 0$ and $X_{\frac{1}{2}}^1(x, n + \frac{1}{2}) = \frac{1}{2}$. Thus, $\rho(\tilde{f}_{\frac{1}{2}}) = \{(0, 0), (\frac{1}{2}, 0)\}$ as desired. \square

We now define our C^ω 3-torus diffeomorphism: Let $\tilde{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by

$$\tilde{F}(x, y, z) = (\tilde{F}_{\tilde{h}(z)}^1(x, y), \tilde{f}_{\tilde{h}(z)}^2(x, y), z),$$

where $\tilde{f}_\lambda = (\tilde{f}_\lambda^1, \tilde{f}_\lambda^2)$ was defined above and $\tilde{h} : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$\tilde{h}(z) = \frac{1}{4} [1 - \cos(2\pi z)].$$

Notice \tilde{F} covers a C^ω map F on \mathbb{T}^3 .

Theorem 3.2. *There exists a C^ω diffeomorphism of \mathbb{T}^3 , isotopic to the identity, with lift $\tilde{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $\rho(\tilde{F})$ is not closed.*

Proof. On each 2-torus, $\{\underline{z} = e^{2\pi i z} = \text{const}\}$, F is the time one map of the $X_{h(z)}$ -flow. Thus F is a C^ω diffeomorphism.

We consider two cases, $z = \frac{1}{2}$ and $z \neq \frac{1}{2}$. When $z = \frac{1}{2}$, $\tilde{h}(z) = \frac{1}{2}$ and by Lemma 3.1, $\rho(\tilde{f}_{\tilde{h}(z)}) = \{(0, 0), (\frac{1}{2}, 0)\}$. So since F is the identity in the third coordinate,

$$\rho(\tilde{F})(x, y, z) = \{(0, 0, 0), (\frac{1}{2}, 0, 0)\}.$$

When $z \neq \frac{1}{2}$, $\tilde{h}(z) \in [0, \frac{1}{2})$. So by Lemma 3.1, $\rho(\tilde{f}_{\tilde{h}(z)}) = (\frac{1}{4}, \tau(\tilde{h}(z)))$. Thus,

$$\rho(\tilde{F})(x, y, z) = \{(\frac{1}{4}, \tau(\tilde{h}(z)), 0)\}.$$

We now combine the two cases. Recall that τ is non-increasing, $\tau(\lambda) > 0$, and $\tau(\lambda) \rightarrow 0$ as $\lambda \rightarrow \frac{1}{2}$. Thus

$$\rho(\tilde{F}) = \{(0, 0, 0)\} \cup \{(\frac{1}{2}, 0, 0)\} \cup \left\{ (\frac{1}{4}, y, 0) \mid 0 < y < \tau(0) \right\},$$

which is not closed. □

Comments: By compactness, $\tau(0) > 0$. A computer computation shows $\tau(0) \approx 1$. However, by rescaling the vector fields $\{X_\lambda\}$, we can arrange for $\tau(0)$ to equal any desired positive real number.

4. EXTREME POINTS OF ROTATION SETS

In this section we prove that rotation sets of lifts of n -torus homeomorphisms ($n \geq 1$) contain their extreme points. We need several theorems from functional analysis.

Theorem 4.1 (Krein-Milman). *Suppose \mathbb{E} is a locally convex topological vector space. A compact convex subset X is the closed convex hull of the set of its extreme points.*

Let $\mathcal{M}_I(\tilde{G})$ denote the set of lifts of regular Borel G -invariant probability measures on \mathbb{T}^n (as defined in Section 2).

Note that $\mathcal{M}_I(\tilde{G})$ is simply the space of \tilde{G} -invariant regular Borel measures on \mathbb{R}^n that are invariant under deck translations and have the property $\tilde{\mu}(Q) = 1$ for each fundamental domain Q .

There is considerable awkwardness in attempting to transport dynamical and measure theory arguments back and forth between a manifold and its covering space. One solution is to make all computations in the covering space. Towards this end, we shall identify the topological vector space \mathcal{S} of signed measures on \mathbb{T}^n with the space $\tilde{\mathcal{S}}$ of their lifted counterparts on \mathbb{R}^n . Topologize the measures via the customary separable and metrizable “weak* topology”: a sequence $\{\tilde{\mu}_n\}$ in $\tilde{\mathcal{S}}$ converges to a measure $\tilde{\mu}$ if and only if the sequence of integrals $\int_Q \tilde{k} d\tilde{\mu}_i$ converges

to $\int_Q \tilde{k} d\tilde{\mu}$ for each lift $\tilde{k} : \mathbb{R}^n \rightarrow \mathbb{R}$ of a map $k : \mathbb{T}^n \rightarrow \mathbb{R}$. By Alaoglu's Theorem, in this topology the space of probability measures is compact (e.g. [D-S], page 424).

We may summarize this discussion as the following lemma:

Lemma 4.2. *Given a torus homeomorphism $G : \mathbb{T}^n \rightarrow \mathbb{T}^n$, the correspondence $\mu \mapsto \tilde{\mu}$ defines a canonical topological vector space isomorphism from the topological vector space \mathcal{S} of signed (regular Borel) measures on \mathbb{T}^n onto the space $\tilde{\mathcal{S}}$ (of lifts of such signed measures) which carries $\mathcal{M}_I(G)$ onto $\mathcal{M}_I(\tilde{G})$.*

A proof of the following lemma may be found in [D-S, p. 428], for example.

Lemma 4.3. *The space of signed measures \mathcal{S} , with the weak* topology is locally convex. Therefore, the space of lifts $\tilde{\mathcal{S}}$ is locally convex.*

Theorem 4.4 (Choquet's Theorem). *A G -invariant Borel probability measure is an extreme point of $\mathcal{M}_I(G)$ if and only if it is ergodic. Hence, the extreme points of $\mathcal{M}_I(\tilde{G})$ are lifts of ergodic measures on \mathbb{T}^n .*

For a simple proof of the preceding theorem in any compact metric space, consult [M, p. 104]. Define the lifted measure $\tilde{\mu}$ to be *ergodic* with respect to \tilde{G} if and only if μ is ergodic with respect to G .

Our next lemma is then a consequence of Theorems 4.1 and 4.4 and Lemmas 4.2 and 4.3.

Lemma 4.5. *The set of \tilde{G} -invariant measures, $\mathcal{M}_I(\tilde{G})$, is the closed convex hull of the (extreme point) set of ergodic \tilde{G} -invariant measures.*

Now suppose again that $\tilde{G} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a lift of an n -torus homeomorphism, G , and let Q be a fundamental domain of \mathbb{R}^n . The following (as in [B-S]) is an easy application of the powerful Birkhoff Ergodic Theorem (for a proof, see [M, p. 89]):

Lemma 4.6. *If $\tilde{\mu} \in \mathcal{M}_I(\tilde{G})$ is ergodic, then*

$$\rho(\tilde{G}, x) = \rho_{\tilde{\mu}}(\tilde{G}) = \int_Q (\tilde{G} - Id) d\tilde{\mu}$$

for $\tilde{\mu}$ -almost all $x \in \mathbb{R}^n \cap Q$.

Proof. Apply the Birkhoff Ergodic Theorem to each component function of the vector field $\tilde{G} - Id$ using the measure $\tilde{\mu}$ and the measure-preserving transformation \tilde{G} . \square

Recall that $\rho_{\text{mes}}(\tilde{G})$ is the union of all mean rotation vectors $\rho_{\tilde{\mu}}(\tilde{G})$.

Lemma 4.7. *The mean rotation set $\rho_{\text{mes}}(\tilde{G})$ is closed, convex and contains the rotation set, $\rho(\tilde{G})$.*

Proof. That $\rho_{\text{mes}}(\tilde{G})$ is closed and convex follows from the fact that $\mathcal{M}_I(\tilde{G})$ is closed and convex in the space of lifted signed measures on \mathbb{R}^n and the map $\mu \mapsto \rho_\mu$ is linear and continuous. Now let $r \in \rho(\tilde{G})(x)$. Then

$$r = \lim_{i \rightarrow \infty} (\tilde{G}^{n_i}(x) - x)/n_i$$

for some subsequence of natural numbers, $n_i \rightarrow \infty$. Let $\tilde{\delta}(x)$ denote the Borel measure that assigns unit mass to each lattice point $x + \vec{n}$ for all $\vec{n} \in \mathbb{Z}^n$. Let $\tilde{\mu}$ be

a limit point of the sequence of measures

$$\tilde{\mu}_i = \sum_{j=1}^{n_i} \frac{1}{n_i} \tilde{\delta}(\tilde{G}^j(x)).$$

Recall that the set $\mathcal{M}_T(\tilde{G})$ of invariant measures is compact (hence, sequentially compact in the metrizable weak* topology). Then $\tilde{\mu}$ is \tilde{G} -invariant and $\lim_{i \rightarrow \infty} \rho_{\tilde{\mu}_i}(\tilde{G}) = \rho_{\tilde{\mu}}(\tilde{G}) = r$. \square

Theorem 4.8. *The rotation set $\rho(\tilde{G})$ contains the extreme points of the mean rotation set, $\rho_{\text{mes}}(\tilde{G})$. In particular, $\rho_{\text{mes}}(\tilde{G})$ equals the closed convex hull of $\rho(\tilde{G})$. Furthermore, $\rho(\tilde{G})$ contains the extreme points of its closure. Finally, these extreme points correspond to ergodic mean rotation numbers.*

Proof. Let r be an extreme point of $\rho_{\text{mes}}(\tilde{G})$. Then $r = \rho_{\tilde{\mu}}(\tilde{G})$ for some \tilde{G} -invariant probability measure μ . By Lemma 4.5, μ lies in the closed convex hull of the \tilde{G} -invariant ergodic measures. In particular, we can write $\mu = \sum t_i \mu_i$, where $\sum t_i = 1$, $0 \leq t_i \leq 1$, and each μ_i is ergodic. (This sum may be infinite.) And $r = \rho_{\tilde{\mu}}(\tilde{G}) = \sum t_i \rho_{\mu_i}(\tilde{G})$.

If μ is not ergodic, we may assume, by reordering the index if necessary, that $0 < t_1 < 1$. In such case define the probability measures $\nu_1 = \mu_1$ and $\nu_2 = \frac{1}{M} \sum_{i \geq 2} t_i \mu_i$, where $M = \sum_{i \geq 2} t_i$. Then $r = \rho_{\tilde{\mu}}(\tilde{G}) = t_1 \rho_{\nu_1} + (1 - t_1) \rho_{\nu_2}$. But, by definition of extreme point, this only happens if $\rho_{\nu_1} = \rho_{\tilde{\mu}} = \rho_{\nu_2}$. So $\rho_{\tilde{\mu}} = \rho_{\tilde{\mu}_1}$, where $\tilde{\mu}_1$ is ergodic and by Lemma 4.6 $\rho_{\tilde{\mu}_1}(\tilde{G}) \in \rho(\tilde{G})$. The last two statements in the theorem follow from Lemma 4.7. \square

5. PSEUDO-ROTATION SETS ARE CLOSED

The pseudo-rotation set $\rho_\psi(\tilde{G}) \supset \rho(\tilde{G})$ ([B-S]). In this section we prove that $\rho_\psi(\tilde{G})$ is closed.

Theorem 5.1. *Let $\tilde{G} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ($n \geq 1$) be the lift of an n -torus homeomorphism. Then the pseudo-rotation set $\rho_\psi(\tilde{G})$ is closed.*

Proof. Assume that

$$r \in \overline{\rho_\psi(\tilde{G})} = \overline{\bigcap_{\epsilon > 0} \rho_\psi(\tilde{G}, \epsilon)}.$$

Then r is in $\overline{\rho_\psi(\tilde{G}, \epsilon)}$ for all $\epsilon > 0$. So for each $\epsilon > 0$, there exist ϵ pseudo-orbits \mathcal{O}_k such that $r = \lim_{k \rightarrow \infty} r_k(\epsilon)$, for some $r_k(\epsilon) \in \rho_\psi(\mathcal{O}_k)$.

We will construct a 2ϵ pseudo-orbit $\{z_m\}$ for \tilde{G} such that $\rho_\psi(\{z_m\}) = r$. Let $Q = [0, 1]^n$ as before. By passing to a subsequence we may assume $\{\overline{\mathcal{O}_{n_k}} \cap Q\}$ converges (in the Hausdorff topology on compact subsets). By periodicity of the lift, the convergence holds for each choice Q of fundamental domain, and, therefore, we can choose k sufficiently large that each vector in \mathcal{O}_{n_k} lies within ϵ of some vector in $\mathcal{O}_{n_{k+1}}$.

Now choose $\{z_0, z_1, \dots, z_{m_1}\} \subset \mathcal{O}_{n_1}$ with m_1 large enough that

$$\left\| \frac{z_{m_1} - z_0}{m_1} - r_{n_1} \right\| < \frac{1}{2}.$$

Choose $z_{m_1+1} \in \mathcal{O}_{n_2}$ such that $\|z_{m_1} - z_{m_1+1}\| < \epsilon$. Next choose $m_2 \gg m_1$ and an ϵ -chain $\{z_{m_1+1}, \dots, z_{m_2}\} \subset \mathcal{O}_{n_2}$ such that

$$\left\| \frac{z_{m_2} - z_0}{m_2} - r_{n_2} \right\| < \frac{1}{4}.$$

Notice $\{z_0, z_1, \dots, z_{m_2}\}$ is a 2ϵ -chain for \tilde{G} .

Continuing recursively, define $\{z_{m_k}\}$ and a 2ϵ pseudo-orbit $\{z_0, z_1, \dots\}$ such that for all k ,

$$\left\| \frac{z_{m_k} - z_0}{m_k} - r_{n_k} \right\| < \frac{1}{2k}.$$

Thus $\rho_\psi(\{z_m\}) = r$ and $r(\epsilon) \in \rho_\psi(\tilde{G}, 2\epsilon)$.

So, $r \in \rho_\psi(\tilde{G})$ as desired. \square

To complete our investigation of the lift \tilde{F} of a 3-torus diffeomorphism constructed in Section 3, we now locate its pseudo-rotation set. An easy proposition is that the pseudo-rotation set of the restriction of \tilde{G} to the lift $\tilde{\Lambda}$ of a chain transitive set Λ in \mathbb{T}^n is closed and convex. The reason is that one can construct pseudo-orbit itineraries for each $\epsilon > 0$ that spend an arbitrary number of iterates on any desired true orbit. On the other hand an ϵ -chain only requires a bounded number of iterates to join any two points of Λ . Thus, we have

Proposition 5.2.

$$\rho_\psi(\tilde{F}) = \langle \rho(\tilde{F}) \rangle.$$

Notes: 1) Notice $\rho(\tilde{F})$ contains its extreme points (as required in Theorem 4.8). But $\rho(\tilde{F})$ does not have any (non-empty) 3-dimensional interior nor the (non-empty) 2-dimensional interior of its convex hull.

2) Consider the Cartesian product of \tilde{F} with the identity map on \mathbb{R}^{n-3} ($n \geq 4$). This map covers a C^ω diffeomorphism of \mathbb{T}^n . And the geometry of its rotation set is identical to that of \tilde{F} .

3) Any compact subset of \mathbb{R}^n can be realized as the rotation set of some diffeomorphism $F : \mathbb{T}^{2n} \rightarrow \mathbb{T}^{2n}$, using methods similar to the construction in this paper.

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