ON A FIXED POINT PROBLEM OF REICH

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(Communicated by Palle E. T. Jorgensen)

Abstract. In this paper, we give an affirmative answer to a fixed point problem of S. Reich.

1. Introduction

Let $X$ be a metric space. $CB(X)$ stands for the set of all non-empty closed bounded subsets of $X$. $CB(X)$ is a metric space with the Hausdorff metric $H$. In [6], S. Reich presented the following

**Problem.** Let $(X, d)$ be a complete metric space. Suppose that $F : X \rightarrow CB(X)$ satisfies $H(F(x), F(y)) \leq K(d(x, y))d(x, y)$ for all $x, y$ in $X$, $x \neq y$, where $K : (0, +\infty) \rightarrow [0, 1)$ and $\lim_{r \to t} sup K(r) < 1$ for all $0 < t < +\infty$. Does $F$ have a fixed point?

In fact, this problem was raised by S. Reich in [4]. S. Reich [5] also gives an affirmative answer to this problem when $Fx$ is non-empty compact for $x \in X$.

In this paper, we give an affirmative answer to this problem. We have the following results.

**Theorem 1.** Let all the conditions of the above problem be satisfied. Then $F$ has a fixed point if and only if there exists a closed subset $Y \subseteq X$, $Fx \cap Y \neq \emptyset$ for all $x \in Y$, such that for each closed subset $Z \subseteq Y$, if $Fx \cap Z \neq \emptyset$ for all $x \in Z$, then $d(x, Fx) = d(x, Fx \cap Z)$, $\forall x \in Z$.

**Remark.** When $F$ is single valued, let $Y = X$; then for each subset $Z \subseteq X$, such that $Fx \in Z$ for all $x \in Z$, we must have $d(x, F(x)) = d(x, F(x) \cap Z)$.

**Theorem 2.** Let $(X, d)$ be a complete metric space. $F : X \rightarrow CB(X)$ satisfies the following conditions.

1. If $Y \subseteq X$ is a non-empty closed bounded subset, and $Fx \cap Y \neq \emptyset$ for all $x \in Y$, then $d(x, Fx) = d(x, Fx \cap Y)$ for all $x \in Y$;
2. $H(F(x), F(y)) \leq K(d(x, y))d(x, y)$, $\forall x, y \in X$, $x \neq y$, where $K : (0, +\infty) \rightarrow [0, 1)$ and $\lim_{r \to t} sup K(r) < 1$ for all $0 < t < +\infty$.

Then $F$ has a fixed point.
2. Proofs

We first prove Theorem 2, then we can prove Theorem 1 similarly to the proof of Theorem 2.

Proof of Theorem 2. Take \( t_n \in (0, +\infty), n = 1, 2, \ldots, \) and \( t_1 > t_2 > \cdots > t_n \to 0. \) Since \( \lim_{r \to t_n^+} \sup K(r) < 1, \) there exist \( 0 \leq k_n < 1 \) and \( \delta_n > 0, \) such that \( K(r) \leq k_n, \forall r \in (t_n, t_n + \delta_n), n = 1, 2, \ldots. \)

Let \( \eta_n = \min\left(\frac{\delta_n}{2}, \frac{1}{n}\right), \) \( \varepsilon_n = t_n + \eta_n; \) then \( K(r) \leq k_n, \forall r \in [\varepsilon_n - \frac{\eta_n}{2}, \varepsilon_n + \eta_n], \) \( n = 1, 2, \ldots. \) It is easy to see \( \varepsilon_n \to 0^+ \) as \( n \to \infty. \)

**Step 1.** For \( \varepsilon_1 > 0, \) we prove there exists \( x_1 \in X, \) such that

\[
(2.1) \quad Fx \cap B_1 \neq \emptyset, \quad \forall x \in B_1 = \{x|d(x, x_1) \leq \varepsilon_1\}.
\]

Suppose (2.1) is not true. Then for each \( x \in X, \) there exists \( x_0 \in X, d(x, x_0) \leq \varepsilon_1, \) but

\[
d(x, y) > \varepsilon_1, \quad \forall y \in Fx_0.
\]

**Case (a).** If \( d(x, x_0) < \varepsilon_1 - \frac{\eta_1}{2}, \) then

\[
d(x, Fx) \geq d(x, Fx_0) - H(Fx_0, Fx) \geq \varepsilon_1 - K(d(x_0, x))d(x_0, x) > \varepsilon_1 - d(x, x_0) > \frac{\eta_1}{2}.
\]

**Case (b).** If \( d(x, x_0) \geq \varepsilon_1 - \frac{\eta_1}{2}, \) then

\[
d(x, Fx) \geq d(x, Fx_0) - H(Fx_0, Fx) \geq \varepsilon_1 - K(d(x_0, x))d(x_0, x) \geq \varepsilon_1 - k_1 \varepsilon_1 = (1 - k_1)\varepsilon_1.
\]

From Cases (a) and (b), we have

\[
(2.2) \quad d(x, Fx) \geq \min\left\{\frac{\eta_1}{2}, (1 - k_1)\varepsilon_1\right\} > 0, \quad \forall x \in X.
\]

Now, fix \( x_0 \in X \) and \( x_1 \in Fx_0. \) Since

\[
d(x_1, Fx_1) \leq H(Fx_0, Fx_1) \leq K(d(x_0, x_1))d(x_0, x_1) < d(x_0, x_1),
\]

there exists \( x_2 \in Fx_1, \) such that

\[
d(x_2, Fx_1) - \frac{1}{2^2} \leq d(x_2, x_1) \leq d(x_0, x_1)
\]

and

\[
d(x_1, x_2) \leq K(d(x_0, x_1))d(x_0, x_1) + \frac{1}{2^2}
\]

By induction, we get \( x_n \in Fx_{n-1}, \) \( n \geq 3, \) such that

\[
d(x_{n-1}, Fx_{n-1}) - \frac{1}{2^n} \leq d(x_{n-1}, x_n) \leq d(x_{n-2}, x_{n-1})
\]

and

\[
d(x_{n-1}, x_n) \leq K(d(x_{n-1}, x_n))d(x_{n-2}, x_{n-1}) + \frac{1}{2^n}.
\]

By the construction of \( \{x_n\}, \) we know \( \lim_{n \to \infty} d(x_{n-1}, x_n) = S_0 \) exists.
Suppose $S_0 > 0$. Then
\[
\lim_{n \to \infty} d(x_{n-1}, x_n) \leq \lim_{n \to \infty} \left[ K(d(x_{n-2}, x_{n-1}))d(x_{n-2}, x_{n-1}) + \frac{1}{2^n} \right] \\
\leq \lim_{n \to \infty} K(r) \lim_{n \to \infty} d(x_{n-2}, x_{n-1}).
\]
So $S_0 \leq \lim_{n \to \infty} K(r) S_0 < S_0$, a contradiction.

Hence we have $\lim_{n \to \infty} d(x_{n-1}, x_n) = 0$. This implies that
\[
\lim_{n \to \infty} d(x_{n-1}, Fx_{n-1}) \leq \lim_{n \to \infty} \left[ d(x_{n-1}, x_n) + \frac{1}{2^n} \right] = 0,
\]
a contradiction to (2.2). So (2.1) is true.

**Step 2.** For $\varepsilon > 0$, we prove there exists $x_2 \in B_1$, such that
\[
Fx \cap B_2 \neq \emptyset, \quad \forall x \in B_2 = \{ x \in B_1 | d(x, x_2) \leq \varepsilon \}.
\]
Suppose (2.3) is not true. For each $x \in B_1$, there exists $y_0 \in B_1$, such that $d(x, y_0) \leq \varepsilon$, but
\[
d(x, y) > \varepsilon, \quad \forall y \in Fy_0.
\]

With the same argument of Cases (a) and (b) in Step 1, we get
\[
d(x, Fx) \geq \min \left\{ \frac{\eta_2}{2}, (1 - k_2)\varepsilon_2 \right\}, \quad \forall x \in B_1.
\]
Now, fix $x_0 \in B_1$, $x_1 \in Fx_0 \cap B_1$. By assumption (1),
\[
d(x_1, Fx_0 \cap B_1) = d(x_1, Fx_1) \leq H(Fx_0, Fx_1) \leq K(d(x_0, x_1))d(x_0, x_1).
\]
So there exists $x_2 \in Fx_1 \cap B_1$, such that
\[
d(x_1, Fx_1) - \frac{1}{2^2} \leq d(x_1, x_2) \leq d(x_0, x_1)
\]
and
\[
d(x_1, x_2) \leq K(d(x_0, x_1))d(x_0, x_1) + 1/2^2.
\]

Generally, we get $x_n \in Fx_{n-1} \cap B_1$, $n \geq 3$, such that
\[
d(x_{n-1}, Fx_{n-1}) - \frac{1}{2^n} \leq d(x_{n-1}, x_n) \leq d(x_{n-2}, x_{n-1})
\]
and
\[
d(x_{n-1}, x_n) \leq K(d(x_{n-2}, x_{n-1}))d(x_{n-2}, x_{n-1}) + \frac{1}{2^n}.
\]
So $\lim_{n \to \infty} d(x_{n-1}, x_n)$ exists and equals zero.

We get $\lim_{n \to \infty} d(x_{n-1}, Fx_{n-1}) = 0$, a contradiction to (2.4). So (2.3) is true.

**Step 3.** By induction, we get $x_{n+1} \in B_n$, such that
\[
Fx \cap B_{n+1} \neq \emptyset, \quad \forall x \in B_{n+1} = \{ x \in B_n | d(x, x_{n+1}) \leq \varepsilon_{n+1} \}, \quad n \geq 2.
\]

It is obvious that $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots \supseteq B_n \supseteq \cdots$, and
\[
\lim_{n \to \infty} \text{diam}(B_n) = 0.
\]
So there exists only one point $x \in \bigcap_{n \geq 1} B_n$, and $x \in Fx$. This completes the proof.
Proof of Theorem 1. Necessity: If \( F \) has a fixed point, let \( Y = \{ x \in X \mid x \in Fx \} \). Then \( Y \neq \emptyset \) is closed and it is the desired subset.

Sufficiency: Suppose \( Y \subseteq X \) is non-empty closed, \( Fx \cap Y \neq \emptyset, \forall x \in Y \), and for each closed subset \( Z \subseteq Y \), if \( Fx \cap Z \neq \emptyset, \forall x \in Z \), then \( d(x, Fx) = d(x, Fx \cap Z) \), \( \forall x \in Z \).

Let \( \{ \varepsilon_n \} \) be as in the proof of Theorem 2.

Step 1. Take \( B_1 = Y \); then \( d(x, F(x)) = d(x, F(x) \cap B_1), \forall x \in B_1 \).

Step 2. With the same argument of Step 2 in the proof of Theorem 2, we get \( x_2 \in B_1 \), such that \( Fx \cap B_2 \neq \emptyset, \forall x \in B_2 = \{ x \in B_1 \mid d(x, x_2) \leq \varepsilon_2 \} \).

By induction, we get \( x_{n+1} \in B_n \), such that \( Fx \cap B_{n+1} \neq \emptyset, \forall x \in B_{n+1} = \{ x \in B_n \mid d(x, x_{n+1}) \leq \varepsilon_{n+1} \}, n \geq 2 \).

So \( \bigcap_{n \geq 1} B_n \) has only one point; it is the fixed point of \( F \).

Acknowledgement

I am grateful to the referee for his (or her) valuable suggestions.

References


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