ON A FIXED POINT PROBLEM OF REICH

CHEN YU-QING

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Abstract. In this paper, we give an affirmative answer to a fixed point problem of S. Reich.

1. Introduction

Let $X$ be a metric space. $CB(X)$ stands for the set of all non-empty closed bounded subsets of $X$. $CB(X)$ is a metric space with the Hausdorff metric $H$. In [6], S. Reich presented the following problem.

**Problem.** Let $(X, d)$ be a complete metric space. Suppose that $F : X \to CB(X)$ satisfies $H(F(x), F(y)) \leq K(d(x, y))d(x, y)$ for all $x, y$ in $X$, $x \neq y$, where $K : (0, +\infty) \to [0, 1)$ and $\lim_{r \to t} sup K(r) < 1$ for all $0 < t < +\infty$. Does $F$ have a fixed point?

In fact, this problem was raised by S. Reich in [4]. S. Reich [5] also gives an affirmative answer to this problem when $F x$ is non-empty compact for $x \in X$.

In this paper, we give an affirmative answer to this problem. We have the following results.

**Theorem 1.** Let all the conditions of the above problem be satisfied. Then $F$ has a fixed point if and only if there exists a closed subset $Y \subseteq X$, $Fx \cap Y \neq \emptyset$ for all $x \in Y$, such that for each closed subset $Z \subseteq Y$, if $Fx \cap Z \neq \emptyset$ for all $x \in Z$, then $d(x, Fx) = d(x, Fx \cap Z)$, $\forall x \in Z$.

Remark. When $F$ is single valued, let $Y = X$; then for each subset $Z \subseteq X$, such that $Fx \in Z$ for all $x \in Z$, we must have $d(x, F(x)) = d(x, F(x) \cap Z)$.

**Theorem 2.** Let $(X, d)$ be a complete metric space. $F : X \to CB(X)$ satisfies the following conditions.

1. If $Y \subseteq X$ is a non-empty closed bounded subset, and $Fx \cap Y \neq \emptyset$ for all $x \in Y$, then $d(x, Fx) = d(x, Fx \cap Y)$ for all $x \in Y$;

2. $H(F(x), F(y)) \leq K(d(x, y))d(x, y)$, $\forall x, y \in X$, $x \neq y$, where $K : (0, +\infty) \to [0, 1)$ and $\lim_{r \to t} sup K(r) < 1$ for all $0 < t < +\infty$.

Then $F$ has a fixed point.
2. Proofs

We first prove Theorem 2, then we can prove Theorem 1 similarly to the proof of Theorem 2.

Proof of Theorem 2. Take \( t_n \in (0, +\infty), n = 1, 2, \ldots \), and \( t_1 > t_2 > \cdots > t_n \to 0 \).
Since \( \lim_{r \to t_n^+} K(r) < 1 \), there exist \( 0 \leq k_n < 1 \) and \( \delta_n > 0 \), such that \( K(r) \leq k_n \), \( \forall r \in (t_n, t_n + \delta_n), n = 1, 2, \ldots \).

Let \( \eta_n = \min \{ \frac{\eta}{m}, \frac{1}{n^2} \} \), \( \varepsilon_n = t_n + \eta_n \); then \( K(r) \leq k_n \), \( \forall r \in [\varepsilon_n - \frac{\eta}{m}, \varepsilon_n + \eta_n] \), \( n = 1, 2, \ldots \). It is easy to see \( \varepsilon_n \to 0^+ \) as \( n \to \infty \).

Step 1. For \( \varepsilon_1 > 0 \), we prove there exists \( x_1 \in X \), such that
\[
(2.1) \quad Fx \cap B_1 \neq \emptyset, \quad \forall x \in B_1 = \{ x | d(x, x_1) \leq \varepsilon_1 \}.
\]

Suppose (2.1) is not true. Then for each \( x \in X \), there exists \( x_0 \in X \), \( d(x, x_0) \leq \varepsilon_1 \), but
\[
d(x, y) > \varepsilon_1, \quad \forall y \in Fx_0.
\]

Case (a). If \( d(x, x_0) < \varepsilon_1 - \frac{\eta}{2} \), then
\[
d(x, Fx) \geq d(x, Fx_0) - H(Fx_0, Fx) \geq \varepsilon_1 - K(d(x_0, x))d(x_0, x)
\]
\[
> \varepsilon_1 - d(x, x_0) > \frac{\eta}{2}.
\]

Case (b). If \( d(x, x_0) \geq \varepsilon_1 - \frac{\eta}{2} \), then
\[
d(x, Fx) \geq d(x, Fx_0) - H(Fx_0, Fx) \geq \varepsilon_1 - K(d(x_0, x))d(x_0, x)
\]
\[
\geq \varepsilon_1 - k_1 \varepsilon_1 = (1 - k_1) \varepsilon_1.
\]

From Cases (a) and (b), we have
\[
(2.2) \quad d(x, Fx) \geq \min \left\{ \frac{\eta}{2}, (1 - k_1) \varepsilon_1 \right\} > 0, \quad \forall x \in X.
\]

Now, fix \( x_0 \in X \) and \( x_1 \in Fx_0 \). Since
\[
d(x_1, Fx_1) \leq H(Fx_0, Fx_1) \leq K(d(x_0, x_1))d(x_0, x_1) < d(x_0, x_1),
\]
there exists \( x_2 \in Fx_1 \), such that
\[
d(x_1, Fx_1) - \frac{1}{2^2} \leq d(x_1, x_2) \leq d(x_0, x_1)
\]
and
\[
d(x_1, x_2) \leq K(d(x_0, x_1))d(x_0, x_1) + \frac{1}{2^2}
\]

By induction, we get \( x_n \in Fx_{n-1}, n \geq 3 \), such that
\[
d(x_{n-1}, Fx_{n-1}) - \frac{1}{2^n} \leq d(x_{n-1}, x_n) \leq d(x_{n-2}, x_{n-1})
\]
and
\[
d(x_{n-1}, x_n) \leq K(d(x_{n-2}, x_{n-1}))d(x_{n-2}, x_{n-1}) + \frac{1}{2^n}.
\]

By the construction of \( \{ x_n \} \), we know \( \lim_{n \to \infty} d(x_{n-1}, x_n) = S_0 \) exists.
Suppose \( S_0 > 0 \). Then
\[
\lim_{n \to \infty} d(x_{n-1}, x_n) \leq \lim_{n \to \infty} \left[ K(d(x_{n-2}, x_{n-1}))d(x_{n-2}, x_{n-1}) + \frac{1}{2^n} \right] \\
\leq \lim_{r \to S_0^-} K(r) \lim_{n \to \infty} d(x_{n-2}, x_{n-1}).
\]
So \( S_0 \leq \lim_{r \to S_0^-} K(r)S_0 < S_0 \), a contradiction.

Hence we have \( \lim_{n \to \infty} d(x_{n-1}, x_n) = 0 \). This implies that
\[
\lim_{n \to \infty} d(x_{n-1}, Fx_{n-1}) \leq \lim_{n \to \infty} \left[ d(x_{n-1}, x_n) + \frac{1}{2^n} \right] = 0,
\]
a contradiction to (2.2). So (2.1) is true.

**Step 2.** For \( \varepsilon > 0 \), we prove there exists \( x_2 \in B_1 \), such that
\[
\tag{2.3}
Fx \cap B_2 \neq \emptyset, \quad \forall x \in B_2 = \{ x \in B_1 | d(x, x_2) \leq \varepsilon \}.
\]
Suppose (2.3) is not true. For each \( x \in B_1 \), there exists \( y_0 \in B_1 \), such that
\[
d(x, y_0) \leq \varepsilon, \quad \text{but}
\]
\[
d(x, y) > \varepsilon, \quad \forall y \in Fy_0.
\]

With the same argument of Cases (a) and (b) in Step 1, we get
\[
\tag{2.4}
d(x, Fx) \geq \min \left\{ \frac{\eta_2}{2}, (1 - k_2)\varepsilon \right\}, \quad \forall x \in B_1.
\]
Now, fix \( x_0 \in B_1 \), \( x_1 \in Fx_0 \cap B_1 \). By assumption (1),
\[
d(x_1, Fx_1 \cap B_1) = d(x_1, Fx_1) \leq H(Fx_0, Fx_1) \leq K(d(x_0, x_1))d(x_0, x_1).
\]
So there exists \( x_2 \in Fx_1 \cap B_1 \), such that
\[
d(x_1, Fx_1) = \frac{1}{2^2} \leq d(x_1, x_2) \leq d(x_0, x_1)
\]
and
\[
d(x_1, x_2) \leq K(d(x_0, x_1))d(x_0, x_1) + 1/2^2.
\]
Generally, we get \( x_n \in Fx_{n-1} \cap B_1 \), \( n \geq 3 \), such that
\[
d(x_{n-1}, Fx_{n-1}) = \frac{1}{2^n} \leq d(x_{n-1}, x_n) \leq d(x_{n-2}, x_{n-1})
\]
and
\[
d(x_{n-1}, x_n) \leq K(d(x_{n-2}, x_{n-1}))d(x_{n-2}, x_{n-1}) + \frac{1}{2^n}.
\]
So \( \lim_{n \to \infty} d(x_{n-1}, x_n) \) exists and equals zero.

We get \( \lim_{n \to \infty} d(x_{n-1}, Fx_{n-1}) = 0 \), a contradiction to (2.4). So (2.3) is true.

**Step 3.** By induction, we get \( x_{n+1} \in B_n \), such that
\[
\tag{2.5}
Fx \cap B_{n+1} \neq \emptyset, \quad \forall x \in B_{n+1} = \{ x \in B_n | d(x, x_{n+1}) \leq \varepsilon_{n+1} \}, \ n \geq 2.
\]

It is obvious that \( B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots \supseteq B_n \supseteq \cdots \), and
\[
\lim_{n \to \infty} \text{diam}(B_n) = 0.
\]
So there exists only one point \( x \in \bigcap_{n \geq 1} B_n \), and \( x \in Fx \). This completes the proof.
Proof of Theorem 1. Necessity: If $F$ has a fixed point, let $Y = \{x \in X | x \in Fx\}$. Then $Y \neq \emptyset$ is closed and it is the desired subset.

Sufficiency: Suppose $Y \subseteq X$ is non-empty closed, $Fx \cap Y \neq \emptyset$, $\forall x \in Y$, and for each closed subset $Z \subseteq Y$, if $Fx \cap Z \neq \emptyset$, $\forall x \in Z$, then $d(x, F(x)) = d(x, Fx \cap Z)$, $\forall x \in Z$.

Let $\{\varepsilon_n\}$ be as in the proof of Theorem 2.

Step 1. Take $B_1 = Y$; then $d(x, F(x)) = d(x, F(x) \cap B_1)$, $\forall x \in B_1$.

Step 2. With the same argument of Step 2 in the proof of Theorem 2, we get $x_2 \in B_1$, such that $Fx \cap B_2 \neq \emptyset$, $\forall x \in B_2 = \{x \in B_1 | d(x, x_2) \leq \varepsilon_2\}$.

By induction, we get $x_{n+1} \in B_n$, such that $Fx \cap B_{n+1} \neq \emptyset$, $\forall x \in B_{n+1} = \{x \in B_n | d(x, x_{n+1}) \leq \varepsilon_{n+1}\}$, $n \geq 2$.

So $\bigcap_{n \geq 1} B_n$ has only one point; it is the fixed point of $F$.

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References


Department of Mathematics, Sichuan University, Chengdu, People’s Republic of China