A QUICK PROOF OF THE CLASSIFICATION OF SIMPLE REAL LIE ALGEBRAS

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Abstract. Élie Cartan’s classification of the simple Lie algebras over \( \mathbb{R} \) is derived quickly from some structure theory over \( \mathbb{R} \) and the classification over \( \mathbb{C} \).

Élie Cartan classified the simple Lie algebras over \( \mathbb{R} \) for the first time in 1914. There have been a number of simplifications in the proof since then, and these are described in [3, p. 537]. All proofs assume the classification over \( \mathbb{C} \) and a certain amount of structure theory over \( \mathbb{R} \). Recent proofs tend to run to 25 pages. Here is a shorter argument.

Theorem. Up to isomorphism, the only simple Lie algebras over \( \mathbb{R} \) that are neither complex nor compact are those in Cartan’s list as organized in [3, p. 518].

We use terminology as in [3]. Let \( g_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0 \) be a Cartan decomposition of a noncomplex simple Lie algebra over \( \mathbb{R} \), and let \( \theta \) be the Cartan involution. Choose a maximal abelian subspace \( t_0 \) of \( \mathfrak{k}_0 \) and extend to a maximally compact Cartan subalgebra \( h_0 = t_0 \oplus a_0 \) of \( g_0 \). Removal of subscripts 0 will indicate complexifications. Let \( \Delta = \Delta(g, h) \) be the root system. Roots are imaginary on \( t_0 \) and real on \( a_0 \). All roots are imaginary-valued or complex on \( h_0 \); there are no real-valued roots. Introduce a positive system \( \Delta^+ \) that takes \( it_0 \) before \( a_0 \). The map \( \theta \) carries roots to roots and permutes the simple roots. The complex simple roots move in two-element orbits, while the imaginary simple roots are fixed. By the Diagram of \((g_0, h_0, \Delta^+)\), we mean the Dynkin diagram of \( \Delta \) with the two-element orbits under \( \theta \) so labeled and with the imaginary roots shaded or not, according as the simple root is noncompact (root vector in \( \mathfrak{p} \)) or compact (root vector in \( \mathfrak{k} \)).

Lemma 1. If \((g_0, h_0, \Delta^+)\) and \((g_0', h_0', (\Delta')^+)\) have the same Diagram, then \( g_0 \) and \( g_0' \) are isomorphic.

Proof. We may assume that the complexifications \((g, h, \Delta^+)\) are the same and that the associated compact forms are the same: \( u_0 = t_0 \oplus i\mathfrak{p}_0 = t_0' \oplus i\mathfrak{p}_0' \). Using the conjugacy of compact forms, the conjugacy of maximal abelian subspaces within them, and the standard construction of a compact form from \( h \), we see that we can normalize root vectors \( X_\alpha, \alpha \in \Delta \), as in Theorem 5.5 of [3, p. 176] and obtain \( u_0 \) from \( \{X_\alpha\} \) as in Theorem 6.3 of [3, p. 181].
First suppose \(a_0 = 0\), so that all roots are imaginary. For \(\alpha\) simple we have
\(\theta X_\alpha = \pm X_\alpha\), the sign being + if \(\alpha\) is compact and — if \(\alpha\) is noncompact. The same formula holds for \(\theta'\). Since \(\mathfrak{h}\) and the \(X_\alpha\)'s for \(\alpha\) simple generate \(\mathfrak{g}\), it follows that
\(\theta = \theta'\), hence that \(\mathfrak{t} = \mathfrak{t}'\) and \(p = p'\). Then \(g_0 = g_0'\) is recovered as \((u_0 \cap \mathfrak{t}) \oplus (u_0 \cap p)\).

If \(a_0 \neq 0\), we may not have \(\theta = \theta'\). For \(\alpha \in \Delta\), write \(g_0 = a_0 X_{a_0}\). Then \(a_0 a_{-a} = 1\) and \(a_0 a_{\theta a} = 1\). Since \(\theta\) maps \(u_0 \cap \text{span}\{X_\alpha, X_{-\alpha}\}\) to \(u_0 \cap \text{span}\{X_{\theta a}, X_{-\theta a}\}\), we see that \(\tilde{a}_\alpha = a_{-\alpha}\). Therefore \(|a_\alpha| = 1\). For each pair of complex simple roots \(\alpha\) and \(\theta a\), choose square roots \(a^{1/2}_\alpha\) and \(a^{1/2}_{\theta a}\) whose product is 1. Similarly write \(\theta' X_\alpha = b_\alpha X_{\theta a}\) with \(|b_\alpha| = 1\), and define \(b^{1/2}_\alpha\) and \(b^{1/2}_{\theta a}\) for \(\alpha\) and \(\theta a\) simple. Define \(H\) and \(H'\) in \(\mathfrak{h} \cap u_0\) by the conditions that \(\alpha(H) = \alpha(H') = 0\) for \(\alpha\) simple imaginary and that \(\exp(1/2 \alpha(H)) = a^{1/2}_\alpha\), \(\exp(1/2 \alpha(H')) = b^{1/2}_\alpha\), and \(\exp(1/2 \theta a(H')) = b^{1/2}_{\theta a}\) if \(\alpha\) and \(\theta a\) are complex simple.

A little computation shows that \(\theta \circ \text{Ad}(\exp 1/2(H - H')) = \text{Ad}(\exp 1/2(H - H')) \circ \theta\), from which it follows that \(\mathfrak{t}' = \text{Ad}(\exp 1/2(H - H'))\mathfrak{t},\ \mathfrak{p}' = \text{Ad}(\exp 1/2(H - H'))\mathfrak{p}\), and \(g_0' = \text{Ad}(\exp 1/2(H - H'))g_0\).

The next step is to identify some pairs of distinct Diagrams that correspond merely to changes of \(\Delta^+\). The argument is inspired by [2]. First let us assume that \(a_0 = 0\), i.e., that the automorphism of \(\Delta\) given by \(\theta\) is the identity. Let \(\Lambda\) be the subset of \(i\mathfrak{t}_0\) where all roots take integer values and where all noncompact roots take odd-integer values. If \(\{\omega_j\}\) is the basis dual to the simple roots, then the sum of those \(\omega_j\) corresponding to the noncompact simple roots is a member of \(\Lambda\). The set \(\Lambda\) is discrete, and we let \(H_0\) be a member of \(\Lambda\) as close to 0 as possible.

**Lemma 2.** If \((\Delta^+)'\) is a positive system that makes \(H_0\) dominant, then there is at most one noncompact simple root, say \(\alpha_i\). If the basis dual to the simple roots of \((\Delta^+)'\) is \(\{\omega_j\}\), then there cannot exist \(i'\) such that \(\langle \omega_i - \omega_{i'}, \omega_{i'} \rangle > 0\).

**Proof.** Since \(H_0\) is in \(\Lambda\) and is dominant, 
\(H_0 = \sum n_j \omega_j\) with all \(n_j\) integers \(\geq 0\). If \(n_i > 0\), then \(H_0 - \omega_i\) is dominant and thus has \(\langle H_0 - \omega_i, \omega_i \rangle \geq 0\) with equality if and only if \(H_0 = \omega_i\). Then \(|H_0 - 2 \omega_i|^2 \leq |H_0|^2\) with equality only if \(H_0 = \omega_i\), and minimality forces \(H_0 = \omega_i\). Now let \(H_0 = \omega_i\). If \(\langle \omega_i - \omega_{i'}, \omega_{i'} \rangle > 0\), then \(|H_0 - 2 \omega_{i'}|^2 < |H_0|^2\), in contradiction to minimality.

When \(a_0 \neq 0\), Lemma 2 is to be applied to the part of \(i\mathfrak{t}_0\) corresponding to the span of the imaginary simple roots. The result is that we can associate to any \(g_0\) at least one Diagram in which at most one imaginary root is shaded.

Now we can read off the possibilities. First suppose that the automorphism of \(\Delta\) is the identity. If all roots are unshaded, then \(g_0\) is the compact form. Otherwise exactly one simple root is shaded. For the classical Dynkin diagrams, let the double line or triple point be at the right end, and let the \(i^{th}\) root be shaded. In \(A_n\), we are led to \(\mathfrak{su}(i, n + 1 - i)\). In \(B_n\), we are led to \(\mathfrak{so}(2i, 2n + 1 - 2i)\). In \(C_n\), we are led to \(\mathfrak{sp}(i, n - i)\) if \(i < n\) and to \(\mathfrak{sp}(n, \mathbb{R})\) if \(i = n\). In \(D_n\), we are led to \(\mathfrak{so}(2i, 2n - 2i)\) if \(i \leq n - 2\) and to \(\mathfrak{so}^*(2n)\) otherwise.

For the exceptional Dynkin diagrams, a little checking that compares the second conclusion of Lemma 2 with the fundamental weights (see [1, pp. 260-275]) shows that \(\alpha_i\) in Lemma 2 has to be a node (endpoint vertex) of the Dynkin diagram. Moreover, in \(G_2\), \(\alpha_i\) has to be the long simple root, while in \(E_8\), it cannot be the node on the short branch. In \(E_6\) two nodes are equivalent by outer automorphism. Thus we obtain at most three Lie algebras for \(E_7\); at most two for \(E_6, E_8, F_4\); and
at most one for $G_2$. These are $E_{II}$, $E_{III}$ for $E_6$, $E_{V}$, $E_{VI}$, $E_{VII}$ for $E_7$; $E_{VIII}$, $E_{IX}$ for $E_8$; $F_1$, $F_{II}$ for $F_4$; and $G$ for $G_2$.

When the automorphism of $\Delta$ is not the identity, the Dynkin diagram is $A_n$, $D_n$, or $E_6$. For $A_n$, there is no imaginary simple root if $n$ is even, and there is one if $n$ is odd. For $n$ even we are led to $\mathfrak{sl}(n+1, \mathbb{R})$, while for $n$ odd we are led to $\mathfrak{sl}(n+1, \mathbb{R})$ if the root is shaded and to $\mathfrak{su}^*(n+1)$ if the root is unshaded. For $D_n$, the first $n-2$ simple roots are imaginary. If all are unshaded, we are led to $\mathfrak{so}(1, 2n-1)$. If the $i^{th}$ simple root is shaded, $i \leq n-2$, we are led to $\mathfrak{so}(2i+1, 2n-2i-1)$. For $E_6$, the triple point and the node on the short branch are imaginary. If neither is shaded, we are led to $E_{IV}$, while if either one is shaded, we are led to $E_{I}$.

Note added in proof. David Vogan has pointed out that any Dynkin diagram marked with an involution and having a subset of its one-element orbits shaded is a Diagram for some $\mathfrak{g}_0$. The proof is in the spirit of Lemma 1. Existence of the exceptional simple real Lie algebras follows.

References

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