A QUICK PROOF OF THE CLASSIFICATION
OF SIMPLE REAL LIE ALGEBRAS

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Abstract. Èlie Cartan’s classification of the simple Lie algebras over \( \mathbb{R} \) is derived quickly from some structure theory over \( \mathbb{R} \) and the classification over \( \mathbb{C} \).

Èlie Cartan classified the simple Lie algebras over \( \mathbb{R} \) for the first time in 1914. There have been a number of simplifications in the proof since then, and these are described in [3, p. 537]. All proofs assume the classification over \( \mathbb{C} \) and a certain amount of structure theory over \( \mathbb{R} \). Recent proofs tend to run to 25 pages. Here is a shorter argument.

**Theorem.** Up to isomorphism, the only simple Lie algebras over \( \mathbb{R} \) that are neither complex nor compact are those in Cartan’s list as organized in [3, p. 518].

We use terminology as in [3]. Let \( g_0 = k_0 \oplus p_0 \) be a Cartan decomposition of a noncomplex simple Lie algebra over \( \mathbb{R} \), and let \( \theta \) be the Cartan involution. Choose a maximal abelian subspace \( t_0 \) of \( k_0 \) and extend to a maximally compact Cartan subalgebra \( h_0 = t_0 \oplus a_0 \) of \( g_0 \). Removal of subscripts 0 will indicate complexifications. Let \( \Delta = \Delta(g, h) \) be the root system. Roots are imaginary on \( t_0 \) and real on \( a_0 \). All roots are imaginary-valued or complex on \( h_0 \); there are no real-valued roots. Introduce a positive system \( \Delta^+ \) that takes \( it_0 \) before \( a_0 \). The map \( \theta \) carries roots to roots and permutes the simple roots. The complex simple roots move in two-element orbits, while the imaginary simple roots are fixed. By the Diagram of \((g_0, h_0, \Delta^+)\), we mean the Dynkin diagram of \( \Delta \) with the two-element orbits under \( \theta \) so labeled and with the imaginary roots shaded or not, according as the simple root is noncompact (root vector in \( p \)) or compact (root vector in \( k \)).

**Lemma 1.** If \((g_0, h_0, \Delta^+)\) and \((g'_0, h'_0, (\Delta')^+)\) have the same Diagram, then \( g_0 \) and \( g'_0 \) are isomorphic.

**Proof.** We may assume that the complexifications \((g, h, \Delta^+)\) are the same and that the associated compact forms are the same: \( u_0 = t_0 \oplus ip_0 = t'_0 \oplus ip'_0 \). Using the conjugacy of compact forms, the conjugacy of maximal abelian subspaces within them, and the standard construction of a compact form from \( h \), we see that we can normalize root vectors \( X_\alpha, \alpha \in \Delta \), as in Theorem 5.5 of [3, p. 176] and obtain \( u_0 \) from \( \{X_\alpha\} \) as in Theorem 6.3 of [3, p. 181].

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3257
First suppose $a_0 = 0$, so that all roots are imaginary. For $\alpha$ simple we have $\theta X_\alpha = \pm X_\alpha$, the sign being $+$ if $\alpha$ is compact and $-$ is noncompact. The same formula holds for $\theta'$. Since $\mathfrak{h}$ and the $X_\alpha$’s for $\alpha$ simple generate $\mathfrak{g}$, it follows that $\theta = \theta'$, hence that $\mathfrak{t} = \mathfrak{t}'$ and $\mathfrak{p} = \mathfrak{p}'$. Then $g_0 = g_0'$ is recovered as $(u_0 \cap \mathfrak{t}) \oplus (u_0 \cap \mathfrak{p})$.

If $a_0 \neq 0$, we may not have $\theta = \theta'$. For $\alpha \in \Delta$, write $\theta X_\alpha = a_\alpha X_\theta a_\alpha$. Then $a_\alpha a_{-\alpha} = 1$ and $a_\alpha a_{\theta a_\alpha} = 1$. Since $\theta$ maps $u_0 \cap \text{span}\{X_\alpha, X_{-\alpha}\}$ to $u_0 \cap \text{span}\{X_{\theta a_\alpha}, X_{-\theta a_\alpha}\}$, we see that $a_\alpha = a_{-\alpha}$. Therefore $|a_\alpha| = 1$. For each pair of complex simple roots $\alpha$ and $\theta a_\alpha$, choose square roots $a^{1/2}_\alpha$ and $a^{1/2}_{\theta a_\alpha}$ whose product is 1. Similarly write $\theta' X_\alpha = b_\alpha X_{\theta a_\alpha}$ with $|b_\alpha| = 1$, and define $b^{1/2}_{\alpha}$ and $b^{1/2}_{\theta a_\alpha}$ for $\alpha$ and $\theta a_\alpha$ simple. Define $H$ and $H'$ in $\mathfrak{h} \cap u_0$ by the conditions that $\alpha(H) = \alpha(H') = 0$ for $\alpha$ simple imaginary and that $\exp(\frac{1}{2}\alpha(H)) = a^{1/2}_\alpha$, $\exp(\frac{1}{2}\theta a_\alpha(H)) = a^{1/2}_{\theta a_\alpha}$, $\exp(\frac{1}{2}\alpha(H')) = b^{1/2}_{\alpha}$, and $\exp(\frac{1}{2}\theta a_\alpha(H')) = b^{1/2}_{\theta a_\alpha}$ if $\alpha$ and $\theta a_\alpha$ are complex simple. A little computation shows that $\theta' \circ \text{Ad}(\exp(\frac{1}{2}(H - H'))) = \text{Ad}(\exp(\frac{1}{2}(H - H'))) \circ \theta$, from which it follows that $\mathfrak{t}' = \text{Ad}(\exp(\frac{1}{2}(H - H'))) \mathfrak{t}$, $\mathfrak{p}' = \text{Ad}(\exp(\frac{1}{2}(H - H'))) \mathfrak{p}$, and $g_0' = \text{Ad}(\exp(\frac{1}{2}(H - H'))) g_0$. \hfill $\square$

The next step is to identify some pairs of distinct Diagrams that correspond merely to changes of $\Delta^+$. The argument is inspired by [2]. First let us assume that $a_0 = 0$, i.e., that the automorphism of $\Delta$ given by $\theta$ is the identity. Let $\Lambda$ be the subset of $i \mathfrak{k}_0$ where all roots take integer values and where all noncompact roots take odd-integer values. If $\{\omega_j\}$ is the basis dual to the simple roots, then the sum of those $\omega_j$ corresponding to the noncompact simple roots is a member of $\Lambda$. The set $\Lambda$ is discrete, and we let $H_0$ be a member of $\Lambda$ as close to 0 as possible.

Lemma 2. If $\Delta^+$ is a positive system that makes $H_0$ dominant, then there is at most one noncompact simple root, say $\alpha_1$. If the basis dual to the simple roots of $(\Delta^+)' = \{\omega_j\}$, then there cannot exist $i'$ such that $\langle \omega_i - \omega_{i'}, \omega_{i'} \rangle > 0$.

Proof. Since $H_0$ is in $\Lambda$ and is dominant, $H_0 = \sum n_j \omega_j$ with all $n_j$ integers $\geq 0$. If $n_i > 0$, then $H_0 - \omega_i$ is dominant and thus has $(H_0 - \omega_i, \omega_i) \geq 0$ with equality if and only if $H_0 = \omega_i$. Then $|H_0 - 2\omega_i|^2 \leq |H_0|^2$ with equality only if $H_0 = \omega_i$, and minimality forces $H_0 = \omega_i$. Now let $H_0 = \omega_i$. If $\langle \omega_i - \omega_{i'}, \omega_{i'} \rangle > 0$, then $|H_0 - 2\omega_{i'}|^2 < |H_0|^2$, in contradiction to minimality. \hfill $\square$

When $a_0 \neq 0$, Lemma 2 is to be applied to the part of $i \mathfrak{k}_0$ corresponding to the span of the imaginary simple roots. The result is that we can associate to any $g_0$ at least one Diagram in which at most one imaginary root is shaded.

Now we can read off the possibilities. First suppose that the automorphism of $\Delta$ is the identity. If all roots are unshaded, then $g_0$ is the compact form. Otherwise exactly one simple root is shaded. For the classical Dynkin diagrams, let the double line or triple point be at the right end, and let the $i^{th}$ root be shaded. In $A_n$, we are led to $\mathfrak{su}(i, n + 1 - i)$. In $B_n$, we are led to $\mathfrak{so}(2i, 2n + 1 - 2i)$. In $C_n$, we are led to $\mathfrak{sp}(i, n - i)$ if $i < n$ and to $\mathfrak{sp}(n, \mathbb{R})$ if $i = n$. In $D_n$, we are led to $\mathfrak{so}(2i, 2n - 2i)$ if $i \leq n - 2$ and to $\mathfrak{so}^*(2n)$ otherwise.

For the exceptional Dynkin diagrams, a little checking that compares the second conclusion of Lemma 2 with the fundamental weights (see [1, pp. 260-275]) shows that $\alpha_i$ in Lemma 2 has to be a node (endpoint vertex) of the Dynkin diagram. Moreover, in $G_2$, $\alpha_1$ has to be the long simple root, while in $E_8$, it cannot be the node on the short branch. In $E_6$ two nodes are equivalent by outer automorphism. Thus we obtain at most three Lie algebras for $E_7$; at most two for $E_6$, $E_8$, $F_4$; and
at most one for $G_2$. These are $E_{II}$, $E_{III}$ for $E_6$, $E_{V}$, $E_{VI}$, $E_{VII}$ for $E_7$; $E_{VIII}$, $E_{IX}$ for $E_8$; $F_{I}$, $F_{II}$ for $F_4$; and $G$ for $G_2$.

When the automorphism of $\Delta$ is not the identity, the Dynkin diagram is $A_n$, $D_n$, or $E_6$. For $A_n$, there is no imaginary simple root if $n$ is even, and there is one if $n$ is odd. For $n$ even we are led to $\mathfrak{sl}(n+1, \mathbb{R})$, while for $n$ odd we are led to $\mathfrak{sl}(n+1, \mathbb{R})$ if the root is shaded and to $\mathfrak{su}^*(n+1)$ if the root is unshaded. For $D_n$, the first $n - 2$ simple roots are imaginary. If all are unshaded, we are led to $\mathfrak{so}(1, 2n - 1)$. If the $i^{th}$ simple root is shaded, $i \leq n - 2$, we are led to $\mathfrak{so}(2i + 1, 2n - 2i - 1)$. For $E_6$, the triple point and the node on the short branch are imaginary. If neither is shaded, we are led to $E_{IV}$, while if either one is shaded, we are led to $E_{I}$.

Note added in proof. David Vogan has pointed out that any Dynkin diagram marked with an involution and having a subset of its one-element orbits shaded is a Diagram for some $\mathfrak{g}_0$. The proof is in the spirit of Lemma 1. Existence of the exceptional simple real Lie algebras follows.

References


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