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# A SIMPLE PROOF OF SINGER'S REPRESENTATION THEOREM

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Abstract. Let $\Omega$ be a compact Hausdorff space and $X$ a Banach space. Singer's theorem states that under the dual pairing $(f, m) \mapsto \int \langle f, dm\rangle$, the dual space of $C(\Omega; X)$ is isometric to $rcabv(\Omega; X')$. Using the Hahn-Banach theorem and the (scalar) Riesz representation theorem, a proof of Singer's theorem is given which appears to be simpler than the proofs supplied earlier by Singer (1957, 1959) and Dinculeanu (1959, 1967).

The notation of the abstract is supposed to be self-explanatory. A Borel measure $m \in cabv(\Omega; X)$ is called regular ($m \in rcabv$) iff its variation $|m|$ is regular. (As a corollary of our approach it is shown below that this is equivalent to the weakest possible notion of regularity.)

*Proof of Singer's representation theorem.* a) Let $m \in cabv(\Omega; X')$; then routine verifications show that $\varphi_m(f) := \int\langle f, dm\rangle$ defines a functional $\varphi_m \in C(\Omega; X)'$ of norm $\|\varphi_m\| \le \|m\|$ $(= |m|(\Omega))$. Under the additional requirement that $\langle x, m\rangle$ be regular $\forall\, x \in X$, $m$ is uniquely determined by $\varphi_m$, by the uniqueness part of the (scalar) Riesz theorem.

b) Conversely, let $\varphi \in C(\Omega; X)'$ be given. Equip the unit ball $B_{X'}$ with the weak* topology and consider the isometric embedding $C(\Omega; X) \hookrightarrow C(\Omega \times B_{X'})$ sending $f$ into the scalar function $(\omega, x') \mapsto \big\langle f(\omega), x'\big\rangle$. The Hahn-Banach theorem combined with the Riesz representation theorem [R, 6.19] produces a complex regular Borel measure $\nu$ on $\Omega \times B_{X'}$ satisfying $\|\nu\| = \|\varphi\|$ and $\forall\, f \in C(\Omega; X) : \varphi(f) = \int_{\Omega\times B_{X'}} \big\langle f(\omega),\, x'\big\rangle\, \nu(d\omega,\, dx')$. In particular, $\forall\, u \in C(\Omega)\,\forall\, x \in X$:

$$\varphi(u \otimes x) = \int_{\Omega\times B_{X'}} u(\omega)\langle x, x'\rangle\, \nu(d\omega,\, dx') = \int_{\Omega} u\, d\mu_x$$

for this Borel measure $\mu_x$ on $\Omega$:

$$\mu_x(A) := \int_{\Omega\times B_{X'}} \mathbf{1}_A(\omega)\langle x,\, x'\rangle\, \nu(d\omega,\, dx').$$

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Obviously

$$\begin{aligned}\left|\mu_x(A)\right| &\le \|x\| \int\limits_{\Omega\times B_{X'}} \mathbf{1}_A(\omega)|\nu|(d\omega,\, dx') \\ &= \|x\| \int\limits_{\Omega} \mathbf{1}_A(\omega)\,\nu_1(d\omega) = \|x\|\,\nu_1(A)\,,\end{aligned}$$

where $\nu_1 := |\nu| \circ \mathrm{pr}_1^{-1}$ is a positive *regular* Borel measure on $\Omega$, as is easily verified. Defining, for Borel sets $A \subset \Omega$ and $x \in X$, $m(A)(x) := \mu_x(A)$, we have immediately $m(A) \in X'$, $\|m(A)\| \le \nu_1(A)$, hence $|m| \le \nu_1$ so that $m \in rcabv(\Omega; X')$ and $\|m\| \le \nu_1(\Omega) = \|\nu\| = \|\varphi\|$. By construction, $m$ represents $\varphi$ on $C(\Omega) \otimes X$ which is dense in $C(\Omega; X)$ [DU, p.225]. Invoke a) to conclude that $m$ represents $\varphi$ on $C(\Omega; X)$. □

The following corollary is proved directly in [S1, Lemma 3].

**Corollary.** *For $m \in cabv(\Omega; Y)$, the following are equivalent:*

1° $|m|$ *is regular.*

2° $\langle m,\, y'\rangle$ *is regular* $\forall\, y' \in X$, *a norming subspace of* $Y'$.

*Proof.* To prove 2° ⇒ 1° , we can assume $Y = X'$. Suppose $\langle x,\, m\rangle$ is regular $\forall\, x \in X$. The functional $\varphi_m \in C(\Omega; X)'$ is represented by an $m' \in rcabv(\Omega; X')$ which implies $m = m'$ as noted in a) above. □

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