FIXED POINTS OF APPROXIMABLE MAPS

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Abstract. We present a simple proof of the Leray-Schauder type theorem for approximable multimaps given recently by Ben-El-Mechaiekh and Idzik. We apply this theorem to obtain a Schaefer type theorem, the Birkhoff-Kellogg type theorems, a Penot type theorem for non-self-maps, and quasi-variational inequalities, all related to compact closed approximable maps.

0. Introduction

Recently, Ben-El-Mechaiekh and Idzik [BI] derived a Leray-Schauder type theorem for approximable multimaps from a matching theorem of Ky Fan. We present, in this paper, a simple proof of their theorem using an earlier fixed point theorem due to Ben-El-Mechaiekh et al. Moreover, we apply their theorem to obtain a Schaefer type theorem, the Birkhoff-Kellogg type theorems, a Penot type theorem for non-self-multimaps, and quasi-variational inequalities with respect to compact closed approximable maps. Finally, we indicate that our method works also for other classes of maps including composites of acyclic maps.

A t.v.s. means a Hausdorff topological vector space. Int, Bd, , and co denote the interior, boundary, closure, and convex hull, respectively.

For subsets X and Y of t.v.s. E and F, respectively, a multimap or map Φ : X → Y is a function from X into the power set of Y with nonempty values. Φ is said to be closed if it has a closed graph Gr(Φ) ⊂ X × Y, and compact if its range Φ(X) is contained in a compact subset of Y.

Given two open neighborhoods U and V of the origin 0 of E and F, respectively, a (U, V)-approximative continuous selection of Φ is a continuous function s : X → Y satisfying

\[ s(x) ∈ (Φ[(x + U) ∩ X] + V) ∩ Y \quad \text{for} \quad x ∈ X. \]

A map Φ : X → Y is said to be approximable if its restriction Φ|K to any compact subset K of X admits a (U, V)-approximative continuous selection for every U and V as above.
For properties and examples of approximable maps, we refer to [BI] and references therein.

1. A Leray-Schauder type theorem

We begin with the following particular form of [P2, Theorem 3]:

**Theorem 1.** Let $X$ be a convex subset of a locally convex t.v.s. $E$ and $\Phi : X \to X$ a compact closed approximable map. Then $\Phi$ has a fixed point $\hat{x} \in X$; that is, $\hat{x} \in \Phi(\hat{x})$.

A slightly particular version of Theorem 1 appeared in [Be2, Theorem 2.4], [BD2, Corollary 3.4], [BD3, Corollary 7.3]. In case $X$ itself is compact, Theorem 1 is given in [BD2, Corollary 3.6], [BD3, Corollary 7.6].

From Theorem 1, we give a simple proof of the following theorem due to Ben-El-Mechaiekh and Idzik [BI]:

**Theorem 2.** Let $X$ be a closed subset of a locally convex t.v.s. $E$ such that $0 \in \text{Int} X$ and $\Phi : X \to E$ a compact closed approximable map. Then either

1. $\Phi$ has a fixed point; or
2. $\lambda x \in \Phi(x)$ for some $\lambda > 1$ and $x \in \text{Bd} X$.

**Proof.** Let $R \subset X$ be defined by

$$ R = \{ x \in X : x \in t\Phi(x) \text{ for some } t \in [0,1] \}, $$

which is nonempty since $0 \in R$. Moreover, it is closed since $\Phi$ is closed. Therefore, $R$ is compact since $\Phi$ is compact.

Suppose that

$$ \text{(LS)} \Phi(y) \cap \{ \lambda y : \lambda > 1 \} = \emptyset \text{ for all } y \in \text{Bd} X. $$

Then $R \cap \text{Bd} X = \emptyset$. Since $X$ is completely regular, there exists a continuous function $r : X \to [0,1]$ such that $r(x) = 1$ for $x \in R$ and $r(x) = 0$ for $x \in \text{Bd} X$.

Let $\Psi : E \to E$ be defined by

$$ \Psi(x) = \begin{cases} r(x)\Phi(x) & \text{if } x \in X, \\ \{0\} & \text{if } x \notin X. \end{cases} $$

Since $\Phi$ is compact and closed, so is $\Psi$. Moreover, $\Psi$ is approximable. In fact, for any compact subset $K$ of $X$, if $s : K \to E$ is a $(U,V)$-approximative continuous selection of $\Phi|_K$, then $rs : K \to E$ is a $(U,V)$-approximative continuous selection of $\Psi|_K$. Therefore, $\Psi$ has a fixed point $x \in E$ by Theorem 1. Since $0 \in \text{Int} X \subset X$, if $x \notin X$ and $x \in \Psi(x) = \{0\}$, we have a contradiction. Hence, $x \in X$. Now $x \in \Psi(x)$ implies $x \in R$ and $r(x) = 1$. Therefore, $x \in \Phi(x)$. This completes our proof.

**Remarks.** 1. We followed the method of Schöneberg [Sö]. Theorem 2 includes [Be1, Theorem 5] and many others. See [BI], [P4].

2. In a sense, Theorem 2 is more general than Theorem 1. As Ben-El-Mechaiekh [Be1, p. 314] noted, Theorem 2 works for a self-map $\Phi : X \to X$ where $X$ is star-shaped with the star-center $0 \in \text{Int} X$. Moreover, Theorem 2 also works for a self-map $\Phi : \overline{X} \to \overline{X}$ where $X$ is a shrinkable subset; that is, $\{0,1\}(\overline{X} - p) \subset \text{Int}(X - p)$ for some $p \in X$. See Klee [K].
2. A Schaefer type theorem

From Theorem 2, we have the following:

**Theorem 3.** Let $E$ be a locally convex t.v.s. and $\Phi : E \to E$ a compact closed approximable map. Then either

1. $\Phi$ has a fixed point; or
2. the set $A = \{ x \in E : x \in t\Phi(x) \text{ for some } t \in (0, 1) \}$ is not bounded.

**Proof.** Suppose that $A$ is bounded. Let $X$ be a bounded neighborhood of 0 such that $A \subset \text{Int } X$. Then no $y \in \text{Bd } X$ satisfies $\lambda y \in \Phi(y)$ for any $\lambda > 1$. Therefore, by Theorem 2, $\Phi$ has a fixed point in $X$.

**Remark.** Theorem 3 was first obtained by Schaefer [Sc1, Sc2] for a completely continuous map $f : E \to E$ on a complete locally convex t.v.s. $E$.

3. The Birkhoff-Kellogg type theorems

As an application of Theorem 2, we have the following generalization of the Birkhoff-Kellogg theorem [BK].

**Theorem 4.** Let $X$ be a closed subset of a locally convex t.v.s. $E$ such that $0 \in \text{Int } X$, and $\Phi : X \to E$ a compact closed approximable map such that $\lambda \Phi(X) \cap X = \emptyset$ for some $\lambda$. Then $\Phi|_{\text{Bd } X}$ has an eigenvalue; that is, $\mu x \in \Phi(x)$ for some $\mu \neq 0$ and $x \in \text{Bd } X$.

**Proof.** Note that $\lambda \neq 0$ and $\lambda \Phi : X \to E$ is a compact closed approximable map. Moreover, $\lambda \Phi$ has no fixed point. Therefore, by Theorem 2, there exist $x \in \text{Bd } X$ and $\mu > 1$ such that $\mu x \in \lambda \Phi(x)$, whence we have $(\lambda^{-1} \mu) x \in \Phi(x)$, where $\lambda^{-1} \mu \neq 0$. This completes our proof.

**Remark.** If $\lambda > 0$ in Theorem 4, then $\Phi|_{\text{Bd } X}$ has an invariant direction (a positive eigenvalue); that is, $\mu x \in \Phi(x)$ for some $\mu > 0$ and $x \in \text{Bd } X$.

From Theorem 4, we obtain

**Theorem 5.** Let $S$ be the unit sphere of a normed vector space $E$ of infinite dimension, and $\Phi : S \to E$ a compact closed approximable map such that $0 \notin \Phi(S)$. Then $\Phi$ has an invariant direction.

**Proof.** Since $E$ is infinite dimensional, by the Dugundji extension theorem, there exists a retraction $r : E \to S$ such that $r(x) = x/\|x\|$ if $\|x\| \geq 1$ and $\|r(x)\| = 1$ if $\|x\| \leq 1$. Let $\Psi = \Phi r : E \to E$. Then $\Psi$ is a compact closed approximable map. Let $B$ be the closed unit ball. Then $\lambda \Psi(B) \cap B = \emptyset$ for some $\lambda > 0$ since $\Psi(B) \subset \Phi(S)$ and $0 \notin \Phi(S)$. Therefore, by Theorem 4 with $X = B$, $\Psi|_S$ has an eigenvalue. Since $\lambda > 0$, this eigenvalue is positive. This completes our proof.

Theorem 5 reduces immediately to the following fixed point theorem:

**Theorem 6.** Let $S$ be the unit sphere of a normed vector space $E$. Then $E$ is of infinite dimension if and only if any compact closed approximable map $\Phi : S \to S$ has a fixed point.
4. Fixed points of non-self-maps

Combining Theorems 1 and 2, we obtain the following fixed point theorem for approximable maps:

**Theorem 7.** Let $X$ be a closed convex subset of a locally convex t.v.s. $E$, and $\Phi : X \to E$ a compact closed approximable map. If $\Phi(Bd X) \subset X$, then $\Phi$ has a fixed point.

**Proof.** If $\text{Int} X = \emptyset$, then $X = Bd X$ and $\Phi : X \to X$ has a fixed point by Theorem 1. If $\text{Int} X \neq \emptyset$, then we may assume $0 \in \text{Int} X$. Now for each $x \in Bd X$, $\Phi(x) \subset X$ implies $\Phi(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$ since $X$ is shrinkable; that is, (LS) holds. Therefore, by Theorem 2, $\Phi$ has a fixed point.

**Remark.** For a compact closed map $\Phi : X \to E$ with convex values, Theorem 7 reduces to Penot [Pe, Proposition 1.4], which contains the particular case for a single-valued continuous map due to Brezis. See [Pe].

5. Quasi-variational or variational inequalities

From Theorem 7, we have the following quasi-variational inequality:

**Theorem 8.** Let $X$ be a closed convex subset of a locally convex t.v.s. $E$, $Y$ a compact subset of $E$, and $f : X \times Y \to \mathbb{R}$ an u.s.c. function. Let $T : X \to Y$ be a closed map such that $T(Bd X) \subset X \cap Y$. Suppose that

(i) the function $M$ defined on $X$ by
$$M(x) = \sup_{y \in T(x)} f(x,y) \quad \text{for} \quad x \in X$$

is l.s.c.; and

(ii) the map $\Phi : X \to Y$ defined on $X$ by
$$\Phi(x) = \{y \in T(x) : f(x,y) = M(x)\} \quad \text{for} \quad x \in X$$

is approximable.

Then there exists an $\hat{x} \in X$ such that
$$\hat{x} \in T \hat{x} \quad \text{and} \quad f(\hat{x}, \hat{x}) = M(\hat{x}).$$

**Proof.** Note that the marginal function $M$ in (i) is actually continuous since $f$ is u.s.c. and $T$ is a compact-valued u.s.c. map, by the well-known result of Berge [Br]. Now, each $\Phi(x)$ is nonempty. Moreover, $\Phi$ is a closed map. In fact, let $(x_\alpha, y_\alpha) \in \text{Gr}(\Phi)$, the graph of $\Phi$, and $(x_\alpha, y_\alpha) \to (x, y)$ in $X \times Y$. Then
$$f(x, y) \geq \liminf_\alpha f(x_\alpha, y_\alpha) = \liminf_\alpha M(x_\alpha) \geq \liminf_\alpha M(x_\alpha) \geq M(x)$$

and, since $\text{Gr}(T)$ is closed in $X \times Y$, $y_\alpha \in T(x_\alpha)$ implies $y \in T(x)$. Hence $(x, y) \in \text{Gr}(\Phi)$. Therefore, $\Phi : X \to E$ is a compact closed approximable map satisfying $\Phi(Bd X) \subset T(Bd X) \subset X$. Hence, by Theorem 7, $\Phi$ has a fixed point $\hat{x} \in X$; that is, $\hat{x} \in T \hat{x}$ and $f(\hat{x}, \hat{x}) = M(\hat{x})$. This completes our proof.

**Remark.** If $T : X \to Y \subset X$, then we can obtain Theorem 8 from Theorem 1 without assuming the closedness of $X$. In this case, Theorem 8 is actually equivalent to Theorem 1. Moreover, in this case, Theorem 8 extends Takahashi [T, Theorem 4], which was applied to prove Fan’s generalizations of fixed point theorems of Schauder and Tychonoff.
The following is an immediate consequence of Theorem 8.

**Theorem 9.** Let \( X \) be a compact convex subset of a locally convex t.v.s., \( f : X \times X \to \mathbb{R} \) a continuous function such that all of the sets
\[
\{ y \in X : f(x, y) = \inf_{y \in X} f(x, y) \}
\]
for \( x \in X \) are (1) convex, (2) contractible, (3) decomposable, or (4) \( \infty \)-proximally connected. Then there exists an \( \hat{x} \in X \) such that
\[
f(\hat{x}, \hat{x}) \leq f(\hat{x}, y) \quad \text{for all } y \in X.
\]

**Proof.** In any case (1)-(4), the map \( \Phi : X \to X \) defined by
\[
\Phi(x) = \{ y \in X : f(x, y) = \inf_{y \in X} f(x, y) \} \quad \text{for } x \in X
\]
is a compact closed approximable map. See [BI]. Now, the conclusion follows from Theorem 8 by putting \( X = Y \) and \( T(x) = X \) for all \( x \in X \).

**Remark.** As in [PC], Theorem 9 can be used to obtain variational or variational-like inequalities due to Hartman-Stampacchia, Browder, Lions-Stampacchia, Mosco, Juberg-Karamardian, Park, Karamardian, Parida-Sahoo-Kumar, Behera-Panda, and Siddiqi-Khalilq-Ansari. For the literature, see [PC].

Finally the approximable map in Theorem 1 can be replaced by an acyclic map [P1, Theorem 7(iii)], a composite of acyclic maps [PSW, Theorem 2(iii)], or a composite of admissible maps [P2, Theorem 3(iii)]. Therefore, following our method, Theorems 2-9 also hold for these classes of maps. In fact, we already have Theorem 8 for acyclic maps [P3, Theorem 2] and for admissible maps [PC, Theorem 2]. Hence, as was noted in [BI], the set in Theorem 9 can be acyclic.

**References**


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