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ON PASTING A_p-WEIGHTS

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ABSTRACT. Theorem 3 gives a condition when two A_p-weights can be "pasted" together to yield another A_p-weight. It is subsequently used in Example 6 to give an example that shows that a necessary condition by Gohberg, Krupnik, and Spitkovsky is not sufficient.

An A_p-weight (cf. [HMW]) is a weight function w on $[0, 2\pi]$ such that Fourier series converge in $L^p(w)$, resp. such that the Hilbert transform is a bounded operator on $L^p(w)$. A_p-weights on contours are related to the boundedness of the Cauchy operator for that contour. Given two A_p-weights $w_1 : [a,b] \to \mathbf{R}^+$ and $w_2 : [a,b] \to \mathbf{R}^+$ (with $a < 0 < b$) the question of when the function $w_1\mathbf{1}_{[a,0)} + w_2\mathbf{1}_{[0,b]}$ is also an A_p-weight is important in operator theory for A_p-weights on contours (cf. [GKS]) and in general for the construction of examples (cf. [CU]) of A_p-weights. In this note we give a necessary and sufficient condition for this to happen (cf. Theorem 3) and show some applications (cf. Remark 4, Example 6, Corollary 8, and Example 9).

Definition 1. For $1 < p < \infty$ a function $w : [a,b] \to \mathbf{R}_0^+$ is called an A_p-*weight* iff

$$K_w := \sup_{I \subseteq [a,b]} \left(\frac{1}{|I|} \int_I w(x)\,dx \right) \left(\frac{1}{|I|} \int_I \frac{1}{(w(x))^{1/(p-1)}}\,dx \right)^{p-1} < \infty,$$

where I denotes an arbitrary subinterval of $[a,b]$. w is called an A_1-*weight* iff

$$K_w := \sup_{I \subseteq [a,b]} \left(\frac{1}{|I|} \int_I w(x)\,dx \right) \operatorname{esssup}_I \frac{1}{w} < \infty.$$

K_w is called the A_p-*constant.*

Remark 2. K_w will always be at least 1 as via Hölder's inequality

$$1 = \frac{1}{|I|} \int_I w^{1/p} \frac{1}{w^{1/p}}\,dx \le \left(\frac{1}{|I|} \int_I w\,dx \right)^{1/p} \left(\frac{1}{|I|} \int_I \frac{1}{w^{1/(p-1)}}\,dx \right)^{(p-1)/p}.$$

A similar argument holds for $p = 1$. □

Theorem 3. *Let $1 < p < \infty$ and let $w_1 : [a,0] \to \mathbf{R}_0^+$ and $w_2 : [0,b] \to \mathbf{R}_0^+$ be A_p-weights. Define*

$$w(x) := \begin{cases} w_1(x), & \text{if } x \in [a,0), \\ w_2(x), & \text{if } x \in [0,b]. \end{cases}$$

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Then $w : [a, b] \to \mathbf{R}_0^+$ *is an* A_p*-weight iff*

$$\limsup_{\varepsilon \to 0} \frac{\int_0^\varepsilon w_2(x)\,dx}{\int_{-\varepsilon}^0 w_1(x)\,dx} < \infty$$

and

$$\liminf_{\varepsilon \to 0} \frac{\int_0^\varepsilon w_2(x)\,dx}{\int_{-\varepsilon}^0 w_1(x)\,dx} > 0.$$

Proof. "$\Rightarrow$": Assume that w is an A_p-weight. Then there is a constant $C > 0$ such that for all $\varepsilon > 0$ with $\varepsilon < |a|, b$ we have

$$\begin{aligned}
C &\geq \left(\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} w(x)\,dx\right)\left(\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \frac{1}{(w(x))^{1/(p-1)}}\,dx\right)^{p-1} \\
&\geq \left(\frac{1}{2\varepsilon}\int_0^{\varepsilon} w_2(x)\,dx\right)\left(\frac{1}{2\varepsilon}\int_{-\varepsilon}^{0} \frac{1}{(w_1(x))^{1/(p-1)}}\,dx\right)^{p-1} \frac{\frac{1}{2\varepsilon}\int_{-\varepsilon}^0 w_1(x)\,dx}{\frac{1}{2\varepsilon}\int_{-\varepsilon}^0 w_1(x)\,dx} \\
&= \left(\frac{1}{2\varepsilon}\int_{-\varepsilon}^0 w_1(x)\,dx\right)\left(\frac{1}{2\varepsilon}\int_{-\varepsilon}^{0} \frac{1}{(w_1(x))^{1/(p-1)}}\,dx\right)^{p-1} \frac{\frac{1}{2\varepsilon}\int_0^\varepsilon w_2(x)\,dx}{\frac{1}{2\varepsilon}\int_{-\varepsilon}^0 w_1(x)\,dx} \\
&\geq \frac{1}{2^p}\frac{\int_0^\varepsilon w_2(x)\,dx}{\int_{-\varepsilon}^0 w_1(x)\,dx}.
\end{aligned}$$

To prove the statement about the limit inferior use the same trick to show that the limit superior of the multiplicative inverse is finite.

"$\Leftarrow$": We can assume there are $C > c > 0$ such that for $\varepsilon > 0$ with $\varepsilon < |a|, b$ we have

$$c < \frac{\int_0^\varepsilon w_2(x)\,dx}{\int_{-\varepsilon}^0 w_1(x)\,dx} < C.$$

Then

$$\begin{aligned}
&\left(\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} w(x)\,dx\right)\left(\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \frac{1}{(w(x))^{1/(p-1)}}\,dx\right)^{p-1} \\
&\quad = \left(\frac{1}{2\varepsilon}\int_{-\varepsilon}^0 w_1(x)\,dx + \frac{1}{2\varepsilon}\int_0^\varepsilon w_2(x)\,dx\right) \\
&\qquad \times \left(\frac{1}{2\varepsilon}\int_{-\varepsilon}^0 \frac{1}{(w_1(x))^{1/(p-1)}}\,dx + \frac{1}{2\varepsilon}\int_0^\varepsilon \frac{1}{(w_2(x))^{1/(p-1)}}\,dx\right)^{p-1} \\
&\quad \leq \max\left\{\frac{1}{\varepsilon}\int_{-\varepsilon}^0 w_1(x)dx, \frac{1}{\varepsilon}\int_0^\varepsilon w_2(x)\,dx\right\} \\
&\qquad \times \max\left\{\frac{1}{\varepsilon}\int_{-\varepsilon}^0 \frac{1}{(w_1(x))^{1/(p-1)}}\,dx, \frac{1}{\varepsilon}\int_0^\varepsilon \frac{1}{(w_2(x))^{1/(p-1)}}\,dx\right\}^{p-1}
\end{aligned}$$

$$\le \max\left\{K_{w_1}, K_{w_2}, \left(\frac{1}{\varepsilon}\int_{-\varepsilon}^{0} w_1(x)\,dx\right)\left(\frac{1}{\varepsilon}\int_{0}^{\varepsilon}\frac{1}{(w_2(x))^{1/(p-1)}}\,dx\right)^{p-1},\right.$$
$$\left.\left(\frac{1}{\varepsilon}\int_{0}^{\varepsilon} w_2(x)\,dx\right)\left(\frac{1}{\varepsilon}\int_{-\varepsilon}^{0}\frac{1}{(w_1(x))^{1/(p-1)}}\,dx\right)^{p-1}\right\}$$
$$\le \max\left\{K_{w_1}, K_{w_2}, \frac{\int_{-\varepsilon}^{0} w_1(x)\,dx}{\int_{0}^{\varepsilon} w_2(x)\,dx}\left(\frac{1}{\varepsilon}\int_{0}^{\varepsilon} w_2(x)\,dx\right)\left(\frac{1}{\varepsilon}\int_{0}^{\varepsilon}\frac{1}{(w_2(x))^{1/(p-1)}}\,dx\right)^{p-1},\right.$$
$$\left.\frac{\int_{0}^{\varepsilon} w_2(x)\,dx}{\int_{-\varepsilon}^{0} w_1(x)\,dx}\left(\frac{1}{\varepsilon}\int_{-\varepsilon}^{0} w_1(x)\,dx\right)\left(\frac{1}{\varepsilon}\int_{-\varepsilon}^{0}\frac{1}{(w_1(x))^{1/(p-1)}}\,dx\right)^{p-1}\right\}$$
$$\le \max\left\{K_{w_1}, K_{w_2}, \frac{1}{c}K_{w_2}, CK_{w_1}\right\}.$$

For intervals I that do not contain 0

$$\left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I \frac{1}{(w(x))^{1/(p-1)}}\,dx\right)^{p-1} \le \max\{K_{w_1}, K_{w_2}\}.$$

For intervals $[d,e]$ with $0 \in [d,e]$ and $|d|$, $e \le \min\{|a|, b\}$ we have with $m := \max\{|d|, e\}$

$$\left(\frac{1}{e-d}\int_d^e w(x)\,dx\right)\left(\frac{1}{e-d}\int_d^e \frac{1}{(w(x))^{1/(p-1)}}\,dx\right)^{p-1}$$
$$\le \left(\frac{1}{e-d}\int_{-m}^{m} w(x)\,dx\right)\left(\frac{1}{e-d}\int_{-m}^{m} \frac{1}{(w(x))^{1/(p-1)}}\,dx\right)^{p-1}$$
$$\le 2^p\left(\frac{1}{2m}\int_{-m}^{m} w(x)\,dx\right)\left(\frac{1}{2m}\int_{-m}^{m} \frac{1}{(w(x))^{1/(p-1)}}\,dx\right)^{p-1}$$
$$\le 2^p \max\left\{K_{w_1}, K_{w_2}, \frac{1}{c}K_{w_2}, CK_{w_1}\right\}.$$

Without loss of generality assume that $|a| \le b$. So far we have that $w|_{[a,|a|]}$ and $w|_{[0,b]}$ are A_p-weights. The only intervals left to consider are intervals I that are not contained in $[a,|a|]$ or $[0,b]$. For such intervals I we have $|I| \ge |a|$ and

$$\left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I \frac{1}{(w(x))^{1/(p-1)}}\,dx\right)^{p-1}$$
$$\le \left(\frac{b-a}{|a|}\right)^p\left(\frac{1}{b-a}\int_a^b w(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b \frac{1}{(w(x))^{1/(p-1)}}\,dx\right)^{p-1}. \qquad \square$$

Remark 4 (I. Spitkovsky). The above obviously shows that on the set of all A_p-weights w on intervals $[0, a_w]$ the relation $w_1 \sim w_2$ iff

$$w(x) := \begin{cases} w_1(-x), & \text{if } x \in [-a_{w_1}, 0],\\ w_2(x), & \text{if } x \in [0, a_{w_2}],\end{cases}$$

is an A_p-weight is an equivalence relation. An immediate application of this is the following improvement of Lemma 1.1 of [GKS]: Let Γ be a contour consisting of a finite number of closed curves and open arcs that satisfy the Carleson condition. Assume there are at most finitely many points of self-intersection $z_1, \ldots, z_m$. For

z_k let $\gamma_{k,1}, \ldots, \gamma_{k,n_k}$ be simple arcs that do not contain z_k and such that for some $\varepsilon > 0$ we have $(\Gamma \cap B_\varepsilon(z_k)) \setminus \bigcup_{j=1}^{n_k} \gamma_{k,j} = \{z_k\}$. Then a weight ρ belongs to $A_p(\Gamma)$ iff

(1) ρ is an A_p-weight on every simple subarc of $\Gamma \setminus \{z_1, \ldots, z_m\}$,

(2) ρ is an A_p-weight on all arcs $\gamma_{k,1} \cup \{z_k\} \cap \gamma_{k,j}$ for $j \in \{2, \ldots, n_k\}$.

This cuts the number of arcs to be checked in (2) from the original $\frac{n_k}{2}(n_k - 1)$ in Lemma 1.1 in [GKS] to $n_k - 1$ at each point z_k. In fact one can easily use Theorem 3 to prove the above result without referring to [GKS] or any results from operator theory. □

Example 5. This example shows that even in very simple cases the new A_p-constant can be close to twice the old A_p-constant. Thus one cannot trivially repeat the gluing process infinitely many times and come up with an A_p-weight once again. We consider $p = 2$ and the weight $w(x) = |x|^\alpha$ with $\alpha \in (-1, 1)$, which can be obtained by pasting $w_1 := w|_{[-1,0]}$ to $w_2 := w|_{[0,1]}$. One checks that $K_{w_1} = K_{w_2} = \frac{1}{1-\alpha^2}$, while for $c, d \in (0, 1]$ with $c = md$ we have

$$\begin{aligned}
&\frac{1}{(c+d)^2} \left(\int_{-c}^{d} x^\alpha \, dx \right) \left(\int_{-c}^{d} x^{-\alpha} \, dx \right) \\
&\quad = \frac{1}{(c+d)^2} \left(\frac{1}{\alpha+1} d^{\alpha+1} + \frac{1}{\alpha+1} c^{\alpha+1} \right) \left(\frac{1}{-\alpha+1} d^{-\alpha+1} + \frac{1}{-\alpha+1} c^{-\alpha+1} \right) \\
&\quad = \frac{1}{1-\alpha^2} \left(\frac{1 + m^{\alpha+1} + m^{-\alpha+1} + m^2}{(1+m)^2} \right),
\end{aligned}$$

with the latter parentheses being close to 2 for m small and α close to 1. □

Example 6. For an A_p-weight let

$$T_z(w) := \{\alpha \in \mathbf{R} : |x - z|^\alpha w(x) \text{ is an } A_p\text{-weight}\}.$$

Since $|\ln|x||$ and $|\ln|x||^{-1}$ are A_p-multipliers for $\alpha \in (-1, p-1)$ the functions $w_1(x) := |x|^\alpha$ and $w_2(x) := |x|^\alpha |\ln|x||$ are A_p-weights on $[-1, 1]$. We clearly have $T_0(w_1) = T_0(w_2)$, in fact even more is true, namely (D. Cruz-Uribe):

$$\{f : f w_1 \in A_p\} = \{f : f w_2 \in A_p\}.$$

However $w(x) := w_1(x)\mathbf{1}_{[-1,0]} + w_2(x)\mathbf{1}_{[0,1]}$ is not an A_p-weight as

$$\begin{aligned}
\frac{\int_0^\varepsilon |x|^\alpha |\ln|x|| \, dx}{\int_0^\varepsilon |x|^\alpha \, dx} &= \frac{\frac{1}{\alpha+1} \varepsilon^{\alpha+1} |\ln(\varepsilon)| + \frac{1}{(\alpha+1)^2} \varepsilon^{\alpha+1}}{\frac{1}{\alpha+1} \varepsilon^{\alpha+1}} \\
&= |\ln(\varepsilon)| + \frac{1}{\alpha+1}.
\end{aligned}$$

Thus the condition $T_0(w_1) = T_0(w_2)$, which is necessary for w to be an A_p-weight (cf. [GKS], Corollary 3.3) is not sufficient. □

Remark 7. It is worth mentioning that analogues of Theorem 3 hold for the pasting of weights on $(-\infty, 0]$ and $[0, \infty)$ (one needs the condition for $\varepsilon \to 0$ and for $\varepsilon \to \infty$), for the pasting of A_1-weights (same condition, or a similar condition for the essential suprema of $\frac{1}{w_1}$ and $\frac{1}{w_2}$ on $(-\varepsilon, 0)$ and $(0, \varepsilon)$) and for doubling measures (work with the measures of the intervals $(0, \varepsilon)$ and $(-\varepsilon, 0)$ rather than the integrals of the weights over the intervals).

Corollary 8 (cf. [CU], Theorem 5.3)**.** *Let $w : [-1,1] \to \mathbf{R}_0^+$ be a function that is a doubling weight/A_p-weight on $[-1,0]$ and on $[0,1]$. If $\frac{w(t)}{w(-t)}$ is bounded above and away from zero, then w is a doubling weight/A_p-weight on $[-1,1]$.* $\square$

Example 9. Corollary 8 was the sharpest pasting condition known so far. In the following we give an example of a pasting in which Corollary 8 fails and Theorem 3 leads to success. Let $w_n : [\sum_{k=1}^n 2^{-k}, \sum_{k=1}^{n+1} 2^{-k}) \to \mathbf{R}_0^+$ be defined by

$$w_n(x) := \begin{cases} |x - \sum_{k=1}^n 2^{-k}|^{1/2}, & \text{if } x \in [\sum_{k=1}^n 2^{-k}, 2^{-n-2} + \sum_{k=1}^n 2^{-k}), \\ |x - \sum_{k=1}^{n+1} 2^{-k}|^{1/2}, & \text{if } x \in [2^{-n-2} + \sum_{k=1}^n 2^{-k}, \sum_{k=1}^{n+1} 2^{-k}). \end{cases}$$

Then $w(x) := w_n(x)$ for $x \in [\sum_{k=1}^n 2^{-k}, \sum_{k=1}^{n+1} 2^{-k})$ is an A_2-weight: First consider

$$\begin{aligned} \int_{1-2^{-n}}^1 w(x)\,dx &= \sum_{k=n}^{\infty} 2 \int_{1-2^{-k}}^{(1-2^{-k})+2^{-k-2}} |x-(1-2^{-k})|^{1/2}\,dx \\ &= 2\sum_{k=n}^{\infty} \int_0^{2^{-k-2}} x^{1/2}\,dx = \frac{1}{6}\sum_{k=n}^{\infty} 2^{-3k/2} = \frac{1}{6(1-2^{-3/2})} 2^{-3n/2}. \end{aligned}$$

Similarly we prove:

$$\int_{1-2^{-n}}^1 \frac{1}{w(x)} dx = C 2^{-n/2}.$$

This proves the A_2-condition for intervals that contain 1. Intervals that only intersect at most two of the original intervals $I_n := [\sum_{k=1}^n 2^{-k}, \sum_{k=1}^{n+1} 2^{-k})$ are taken care of by Theorem 3 and intervals I that intersect more than two contain one of the I_n and we can do a crude estimate replacing the integration over I with an integration over $I_{n-2} \cup I_{n-1} \cup I_n \cup \cdots$ to get the A_p-bound. We also see from the above computation that w can be pasted with the A_2-weight $v(x) := |x-1|^{1/2}$ on the interval $[1,2]$ to yield an A_2-weight on $[0,2]$. However it is also clear that the quotient $\frac{w(1-x)}{v(1+x)}$ is not bounded away from zero. $\square$

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References

[BS1] A. Böttcher and I. Spitkovsky, *Wiener-Hopf integral operators with PC symbols on spaces with Muckenhoupt weight*, Rev. Mat. Iberoamericana **9** (1993), 257–279. MR **94e:**45004

[BS2] ———, *On a theorem of Rooney concerning the spectrum of the singular integral operator*, Z. Anal. Anwendungen **12** (1993), 93–96. MR **94h:**47093

[CU] D. Cruz-Uribe, *Piecewise monotonic doubling measures*, Ph. D. dissertation.

[GKS] I. Gohberg, N. Krupnik, I. Spitkovsky, *Banach algebras of singular integral operators with piecewise continuous coefficients.* General contour and weight, Integral Equations Operator Theory **17** (1993), 322–337. MR **94f:**47057

[HMW] R. Hunt, B. Muckenhoupt, R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans.Amer. Math. Soc. **176** (1973), 227–251. MR **47:**701

[JN] R. Johnson, C. Neugebauer, *Change of variable results for A_p-and reverse Hölder RH_r-classes*, Trans. Amer. Math. Soc. **328** (1991), 639–666. MR **92c:**42019

[Sp] I. Spitkovsky, *Singular integral operators with PC symbols on the spaces with general weights*, J. Funct. Anal. **105** (1992), 129–143. MR **93d:**47057

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