ON PASTING $A_p$-WEIGHTS

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ABSTRACT. Theorem 3 gives a condition when two $A_p$-weights can be “pasted” together to yield another $A_p$-weight. It is subsequently used in Example 6 to give an example that shows that a necessary condition by Gohberg, Krupnik, and Spitkovsky is not sufficient.

An $A_p$-weight (cf. [HMW]) is a weight function $w$ on $[0, 2\pi]$ such that Fourier series converge in $L^p(w)$, resp. such that the Hilbert transform is a bounded operator on $L^p(w)$. $A_p$-weights on contours are related to the boundedness of the Cauchy operator for that contour. Given two $A_p$-weights $w_1 : [a, b] \to \mathbb{R}^+$ and $w_2 : [0, b] \to \mathbb{R}^+$ (with $a < 0 < b$) the question of when the function $w_1|_{[a, 0]} + w_2|_{[0, b]}$ is also an $A_p$-weight is important in operator theory for $A_p$-weights on contours (cf. [GKS]) and in general for the construction of examples (cf. [CU]) of $A_p$-weights. In this note we give a necessary and sufficient condition for this to happen (cf. Theorem 3) and show some applications (cf. Remark 4, Example 6, Corollary 8, and Example 9).

Definition 1. For $1 < p < \infty$ a function $w : [a, b] \to \mathbb{R}_+^+$ is called an $A_p$-weight iff

$$K_w := \sup_{I \subseteq [a, b]} \left( \frac{1}{|I|} \int_I w(x) \, dx \right) \left( \frac{1}{|I|} \int_I \left( \frac{1}{w(x)} \right)^{1/(p-1)} \, dx \right)^{p-1} < \infty,$$

where $I$ denotes an arbitrary subinterval of $[a, b]$. $w$ is called an $A_1$-weight iff

$$K_w := \sup_{I \subseteq [a, b]} \left( \frac{1}{|I|} \int_I w(x) \, dx \right) \operatorname{esssup}_I \frac{1}{w} < \infty.$$

$K_w$ is called the $A_p$-constant.

Remark 2. $K_w$ will always be at least 1 as via Hölder’s inequality

$$1 = \frac{1}{|I|} \int_I w^{1/p} \frac{1}{w^{1/p}} \, dx \leq \left( \frac{1}{|I|} \int_I w \, dx \right)^{1/p} \left( \frac{1}{|I|} \int_I \left( \frac{1}{w} \right)^{1/(p-1)} \, dx \right)^{(p-1)/p}.$$

A similar argument holds for $p = 1$.

Theorem 3. Let $1 < p < \infty$ and let $w_1 : [a, 0] \to \mathbb{R}_+^+$ and $w_2 : [0, b] \to \mathbb{R}_+^+$ be $A_p$-weights. Define

$$w(x) := \begin{cases} w_1(x), & \text{if } x \in [a, 0), \\ w_2(x), & \text{if } x \in [0, b]. \end{cases}$$
Then \( w : [a, b] \to \mathbb{R}_0^+ \) is an \( A_p \)-weight iff
\[
\limsup_{\epsilon \to 0} \frac{\int_0^\epsilon w_2(x) \, dx}{\int_{-\epsilon}^0 w_1(x) \, dx} < \infty
\]
and
\[
\liminf_{\epsilon \to 0} \frac{\int_0^\epsilon w_2(x) \, dx}{\int_{-\epsilon}^0 w_1(x) \, dx} > 0.
\]

Proof. \( \Rightarrow \): Assume that \( w \) is an \( A_p \)-weight. Then there is a constant \( C > 0 \) such that for all \( \epsilon > 0 \) with \( \epsilon < |a|, b \) we have
\[
C \geq \left( \frac{1}{2\epsilon} \int_{-\epsilon}^\epsilon w(x) \, dx \right) \left( \frac{1}{2\epsilon} \int_{-\epsilon}^\epsilon \frac{1}{(w(x))^{1/(p-1)}} \, dx \right)^{p-1}
\]
\[
\geq \left( \frac{1}{2\epsilon} \int_{-\epsilon}^0 w_2(x) \, dx \right) \left( \frac{1}{2\epsilon} \int_{-\epsilon}^0 \frac{1}{(w_1(x))^{1/(p-1)}} \, dx \right)^{p-1} \frac{1}{2\epsilon} \int_{-\epsilon}^0 w_1(x) \, dx
\]
\[
= \left( \frac{1}{2\epsilon} \int_{-\epsilon}^0 w_1(x) \, dx \right) \left( \frac{1}{2\epsilon} \int_{-\epsilon}^0 \frac{1}{(w_1(x))^{1/(p-1)}} \, dx \right)^{p-1} \frac{1}{2\epsilon} \int_{-\epsilon}^0 w_2(x) \, dx
\]
\[
\geq \frac{1}{2p} \int_{-\epsilon}^0 w_2(x) \, dx.
\]
To prove the statement about the limit inferior use the same trick to show that the limit superior of the multiplicative inverse is finite.

\( \Leftarrow \): We can assume there are \( C > c > 0 \) such that for \( \epsilon > 0 \) with \( \epsilon < |a|, b \) we have
\[
c < \frac{\int_0^\epsilon w_2(x) \, dx}{\int_{-\epsilon}^0 w_1(x) \, dx} < C.
\]
Then
\[
\left( \frac{1}{2\epsilon} \int_{-\epsilon}^\epsilon w(x) \, dx \right) \left( \frac{1}{2\epsilon} \int_{-\epsilon}^\epsilon \frac{1}{(w(x))^{1/(p-1)}} \, dx \right)^{p-1}
\]
\[
= \left( \frac{1}{2\epsilon} \int_{-\epsilon}^0 w_1(x) \, dx + \frac{1}{2\epsilon} \int_0^\epsilon w_2(x) \, dx \right)
\]
\[
\times \left( \frac{1}{2\epsilon} \int_{-\epsilon}^0 \frac{1}{(w_1(x))^{1/(p-1)}} \, dx + \frac{1}{2\epsilon} \int_0^\epsilon \frac{1}{(w_2(x))^{1/(p-1)}} \, dx \right)^{p-1}
\]
\[
\leq \max \left\{ \frac{1}{\epsilon} \int_{-\epsilon}^0 w_1(x) \, dx, \frac{1}{\epsilon} \int_0^\epsilon w_2(x) \, dx \right\}
\]
\[
\times \max \left\{ \frac{1}{\epsilon} \int_{-\epsilon}^0 \frac{1}{(w_1(x))^{1/(p-1)}} \, dx, \frac{1}{\epsilon} \int_0^\epsilon \frac{1}{(w_2(x))^{1/(p-1)}} \, dx \right\}^{p-1}
\]
ON PASTING $A_p$-WEIGHTS

For intervals $I$ that do not contain $0$

\[
\left( \frac{1}{|I|} \int_I w(x) \, dx \right) \left( \frac{1}{|I|} \int_I \frac{1}{(w(x))^{1/(p-1)}} \, dx \right)^{p-1} \leq \max\{K_{w_1}, K_{w_2}\}.
\]

For intervals $[d, e]$ with $0 \in [d, e]$ and $|d|, e \leq \min\{|a|, b\}$ we have with $m := \max\{|d|, e\}$

\[
\left( \frac{1}{e - d} \int_d^e w(x) \, dx \right) \left( \frac{1}{e - d} \int_d^e \frac{1}{(w(x))^{1/(p-1)}} \, dx \right)^{p-1} \leq \left( \frac{1}{e - d} \int_m^m w(x) \, dx \right) \left( \frac{1}{e - d} \int_m^m \frac{1}{(w(x))^{1/(p-1)}} \, dx \right)^{p-1} \leq 2^p \left( \frac{1}{2m} \int_{-m}^m w(x) \, dx \right) \left( \frac{1}{2m} \int_{-m}^m \frac{1}{(w(x))^{1/(p-1)}} \, dx \right)^{p-1} \leq 2^p \max\{K_{w_1}, K_{w_2}, 1 / e \cdot K_{w_2}, C K_{w_1}\}.
\]

Without loss of generality assume that $|a| \leq b$. So far we have that $w|_{[a, |a|]}$ and $w|_{[0, b]}$ are $A_p$-weights. The only intervals left to consider are intervals $I$ that are not contained in $[a, |a|]$ or $[0, b]$. For such intervals $I$ we have $|I| \geq |a|$ and

\[
\left( \frac{1}{|I|} \int_I w(x) \, dx \right) \left( \frac{1}{|I|} \int_I \frac{1}{(w(x))^{1/(p-1)}} \, dx \right)^{p-1} \leq \left( \frac{b - a}{|a|} \right)^p \left( \frac{1}{b - a} \int_a^b w(x) \, dx \right) \left( \frac{1}{b - a} \int_a^b \frac{1}{(w(x))^{1/(p-1)}} \, dx \right)^{p-1} \cdot \quad \Box
\]

Remark 4 (I. Spitkovsky). The above obviously shows that on the set of all $A_p$-weights $w$ on intervals $[0, a_w]$ the relation $w_1 \sim w_2$ iff

\[
w(x) := \begin{cases}
w_1(x), & x \in [-a_w, 0], \\
w_2(x), & x \in [0, a_w],
\end{cases}
\]

is an $A_p$-weight is an equivalence relation. An immediate application of this is the following improvement of Lemma 1.1 of [GKS]: Let $\Gamma$ be a contour consisting of a finite number of closed curves and open arcs that satisfy the Carleson condition. Assume there are at most finitely many points of self-intersection $z_1, \ldots, z_m$. For
Let $\gamma_{k,1}, \ldots, \gamma_{k,n_k}$ be simple arcs that do not contain $z_k$ and such that for some $\varepsilon > 0$ we have $(\Gamma \cap B_\varepsilon(z_k)) \cup \bigcup_{j=1}^{n_k} \gamma_{k,j} = \{z_k\}$. Then a weight $\rho$ belongs to $A_p(\Gamma)$ iff

1. $\rho$ is an $A_p$-weight on every simple subarc of $\Gamma \setminus \{z_1, \ldots, z_m\}$,
2. $\rho$ is an $A_p$-weight on all arcs $\gamma_{k,1} \cup \{z_k\} \cap \gamma_{k,j}$ for $j \in \{2, \ldots, n_k\}$.

This cuts the number of arcs to be checked in (2) from the original $\frac{n_k}{2}(n_k - 1)$ in Lemma 1.1 in [GKS] to $n_k - 1$ at each point $z_k$. In fact one can easily use Theorem 3 to prove the above result without referring to [GKS] or any results from operator theory. \hfill \Box

**Example 5.** This example shows that even in very simple cases the new $A_p$-constant can be close to twice the old $A_p$-constant. Thus one cannot trivially repeat the gluing process infinitely many times and come up with an $A_p$-weight once again. We consider $p = 2$ and the weight $w(x) = |x|^\alpha$ with $\alpha \in (-1, 1)$, which can be obtained by pasting $w_1 := w_{[-1,0]}$ to $w_2 := w_{[0,1]}$. One checks that $K_{w_1} = K_{w_2} = \frac{1}{1-\alpha^2}$, while for $c, d \in (0, 1]$ with $c = md$ we have

$$\frac{1}{(c+d)^2} \left( \int_{-\varepsilon}^{\varepsilon} x^\alpha \, dx \right) \left( \int_{-\varepsilon}^{\varepsilon} x^{-\alpha} \, dx \right) = \frac{1}{1-\alpha^2} \left( \frac{1}{\alpha+1} d^{\alpha+1} + \frac{1}{\alpha+1} e^{\alpha+1} \right) \left( \frac{1}{\alpha-1} d^{-\alpha+1} + \frac{1}{\alpha+1} e^{-\alpha+1} \right)$$

$$= \frac{1}{1-\alpha^2} \left( \frac{1 + m^{\alpha+1} + m^{-\alpha+1} + m^2}{1+m^2} \right),$$

with the latter parentheses being close to 2 for $m$ small and $\alpha$ close to 1. \hfill \Box

**Example 6.** For an $A_p$-weight let

$$T_z(w) := \{ \alpha \in \mathbb{R} : |x-z|^\alpha w(x) \text{ is an } A_p\text{-weight} \}.$$

Since $|\ln |x||$ and $|\ln |x||^{-1}$ are $A_p$-multipliers for $\alpha \in (-1, p-1)$ the functions $w_1(x) := |x|^\alpha$ and $w_2(x) := |x|^\alpha |\ln |x||$ are $A_p$-weights on $[-1, 1]$. We clearly have $T_0(w_1) = T_0(w_2)$, in fact even more is true, namely (D. Cruz-Uribe):

$$\{ f : f w_1 \in A_p \} = \{ f : f w_2 \in A_p \}.$$

However $w(x) := w_1(x) \mathbf{1}_{[-1,0]} + w_2(x) \mathbf{1}_{[0,1]}$ is not an $A_p$-weight as

$$\frac{\int_0^\varepsilon |x|^{\alpha} |\ln |x|| \, dx}{\int_0^\varepsilon |x|^{\alpha} \, dx} = \frac{\frac{1}{\alpha+1} \varepsilon^{\alpha+1} |\ln (\varepsilon)| + \frac{1}{(\alpha+1)^{\alpha+1}} e^{\alpha+1}}{\frac{1}{\alpha+1} \varepsilon^{\alpha+1}} = |\ln (\varepsilon)| + \frac{1}{\alpha+1}.$$

Thus the condition $T_0(w_1) = T_0(w_2)$, which is necessary for $w$ to be an $A_p$-weight (cf. [GKS], Corollary 3.3) is not sufficient. \hfill \Box

**Remark 7.** It is worth mentioning that analogues of Theorem 3 hold for the pasting of weights on $(-\infty, 0]$ and $[0, \infty)$ (one needs the condition for $\varepsilon \to 0$ and for $\varepsilon \to \infty$), for the pasting of $A_1$-weights (same condition, or a similar condition for the essential supremum of $\frac{1}{\alpha}$ and $\frac{1}{\alpha}$ on $(-\varepsilon, 0)$ and $(0, \varepsilon)$) and for doubling measures (work with the measures of the intervals $(0, \varepsilon)$ and $(-\varepsilon, 0)$ rather than the integrals of the weights over the intervals).
Corollary 8 (cf. [CU], Theorem 5.3). Let \( w : [-1, 1] \rightarrow \mathbb{R}^+_0 \) be a function that is a doubling weight/A\(_p\)-weight on \([-1, 0]\) and on \([0, 1]\). If \( \frac{w(t)}{w(-t)} \) is bounded above and away from zero, then \( w \) is a doubling weight/A\(_p\)-weight on \([-1, 1]\). \( \square \)

Example 9. Corollary 8 was the sharpest pasting condition known so far. In the following we give an example of a pasting in which Corollary 8 fails and Theorem 3 leads to success. Let \( w_n : [\sum_{k=1}^{n} 2^{-k}, \sum_{k=1}^{n+1} 2^{-k}) \rightarrow \mathbb{R}^+_0 \) be defined by

\[
 w_n(x) := \begin{cases} 
    |x - \sum_{k=1}^{n} 2^{-k}|^{1/2}, & \text{if } x \in [\sum_{k=1}^{n} 2^{-k}, 2^{-n-2} + \sum_{k=1}^{n} 2^{-k}), \\
    |x - \sum_{k=1}^{n+1} 2^{-k}|^{1/2}, & \text{if } x \in [2^{-n-2} + \sum_{k=1}^{n} 2^{-k}, \sum_{k=1}^{n+1} 2^{-k}).
\end{cases}
\]

Then \( w(x) := w_n(x) \) for \( x \in [\sum_{k=1}^{n} 2^{-k}, \sum_{k=1}^{n+1} 2^{-k}) \) is an \( A_2 \)-weight: First consider

\[
\int_{1-2^{-n}}^{1} w(x) \, dx = \sum_{k=n}^{\infty} 2 \int_{1-2^{-k}}^{(1-2^{-k}) + 2^{-k-2}} |x - (1-2^{-k})|^{1/2} \, dx \\
= 2 \sum_{k=n}^{\infty} \int_{0}^{2^{-k-2}} x^{1/2} \, dx = \frac{1}{6} \sum_{k=n}^{\infty} 2^{-3k/2} = \frac{1}{6(1-2^{-3/2})} 2^{-3n/2}.
\]

Similarly we prove:

\[
\int_{1-2^{-n}}^{1} \frac{1}{w(x)} \, dx = C 2^{-n/2}.
\]

This proves the \( A_2 \)-condition for intervals that contain 1. Intervals that only intersect at most two of the original intervals \( I_n := [\sum_{k=1}^{n} 2^{-k}, \sum_{k=1}^{n+1} 2^{-k}) \) are taken care of by Theorem 3 and intervals \( I \) that intersect more than two contain one of the \( I_n \) and we can do a crude estimate replacing the integration over \( I \) with an integration over \( I_n \cup I_{n-1} \cup I_n \cup \cdots \) to get the \( A_p \)-bound. We also see from the above computation that \( w \) can be pasted with the \( A_2 \)-weight \( v(x) := |x - 1|^{1/2} \) on the interval \([1, 2]\) to yield an \( A_2 \)-weight on \([0, 2]\). However it is also clear that the quotient \( \frac{w(1-x)}{w(1+x)} \) is not bounded away from zero. \( \square \)

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