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TYCHONOFF'S THEOREM IN A CATEGORY

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ABSTRACT. A categorical proof of Tychonoff's Theorem on the productivity of compact topological spaces is provided.

1. Introduction

In a category with products, a factorization system and a closure operator c, an object X is c-compact if every projection $X \times Y \to Y$ preserves the closure. A product theorem for c-compact objects is provided which gives the Tychonoff Theorem for compact spaces when applied to the category $\mathcal{T}op$ of topological spaces, and the Frolík-Bourbaki Theorem for proper maps when applied to the slices of $\mathcal{T}op$.

2. The categorical setting

We consider a category $\mathcal{X}$ with direct products and an $(\mathcal{E}, \mathcal{M})$-factorization system for morphisms (cf. [7]), for a class $\mathcal{M}$ of monomorphisms in $\mathcal{X}$. The image $f(m)$ of an $\mathcal{M}$-morphism $m : M \to X$ along a morphism $f : X \to Y$ is simply the $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of $f \cdot m$. It is assumed that $\mathcal{E}$ is a *surjectivity class*, in the sense that there is a class $\mathcal{P}$ of objects in $\mathcal{X}$ such that any morphism f of $\mathcal{X}$ belongs to $\mathcal{E}$ if and only if every $P \in \mathcal{P}$ is projective with respect to f, i.e., every map $\mathcal{X}(P, f) : \mathcal{X}(P, X) \to \mathcal{X}(P, Y)$ is surjective.

Furthermore, we assume that $\mathcal{X}$ comes equipped with a *closure operator* $c = (c_X)_{X \in \mathcal{X}}$ with respect to $\mathcal{M}$ in the sense of [3], so that for every $m : M \to X$ in $\mathcal{M}$ one has a morphism $c_X(m) : c_X(M) \to X$ in $\mathcal{M}$, subject to the following conditions: 1. $m \leq c_X(m)$, 2. $(m \leq n \;\Rightarrow\; c_X(m) \leq c_X(n))$, 3. $f(c_X(m)) \leq c_Y(f(m))$, for all $m, n \in \mathcal{M}$ and morphisms $f : X \to Y$. (Here $m \leq n$ means that there exists a morphism k with $n \cdot k = m$. Observe that there is no idempotency requirement.) If a morphism $f : X \to Y$ satisfies the condition $c_Y(f(m)) \leq f(c_X(m))$ for every $m : M \to X$ in $\mathcal{M}$, then it is called *c-preserving*.

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An object $X \in \mathcal{X}$ is called *c-compact* if the projection $X \times Y \to Y$ is c-preserving for every object $Y \in \mathcal{X}$.

3. The Theorem

For objects $X_i \in \mathcal{X}$, $i \in I$, and any subset $J \subseteq I$, let $X^J := \prod_{i \in J} X_i$ and consider the obvious projection $p^I_J : X^I \to X^J$. The closure operator c is said to satisfy the *finite structure property of products* (FSPP) if for all $n : N \to X^I$, $m : M \to X^I$ in $\mathcal{M}$ one has $n \leq c_{X^I}(m)$ whenever $p^I_F(n) \leq c_{X^F}(p^I_F(m))$ for every finite subset $F \subseteq I$ (cf. [5]).

Using the Axiom of Choice, we are now able to prove:

Theorem. *For a closure operator c satisfying FSPP, the direct product of c-compact objects is c-compact.*

Proof. We assume I to be of the form $I = \kappa = \{\beta : 0 \leq \beta < \kappa\}$ for an ordinal $\kappa \geq 1$, and we must prove that the projection $\pi : X^I \to X_0$ is c-preserving for c-compact objects X_α $(0 < \alpha < \kappa)$ and any object X_0. For $0 \leq \beta < \alpha \leq \gamma \leq \kappa$, one has projections $p^\gamma_\alpha : X^\gamma \to X^\alpha$ and $q^\alpha_\beta : X^\alpha \to X_\beta$ such that

$$p^\gamma_\alpha \cdot p^\kappa_\gamma = p^\kappa_\alpha,\ p^\alpha_\alpha = 1,\ \text{and}\ q^\alpha_\beta \cdot p^\gamma_\alpha = q^\gamma_\beta.$$

In this setting, $\pi = p^\kappa_1 = q^\kappa_0$. For $m : M \to X^\kappa$ in $\mathcal{M}$, let $k_\alpha := c_{X^\alpha}(p^\kappa_\alpha(m)) : K_\alpha \to X^\alpha$. From condition 3 for the closure operator c applied to $f := p^\gamma_\alpha$ one obtains morphisms $v^\gamma_\alpha : K_\gamma \to K_\alpha$ with

$$v^\gamma_\alpha \cdot v^\kappa_\gamma = v^\kappa_\alpha,\ v^\alpha_\alpha = 1,\ \text{and}\ k_\alpha \cdot v^\gamma_\alpha = p^\gamma_\alpha \cdot k_\gamma.$$

Our goal is to show that $v^\kappa_1 : K_\kappa = c_{X^\kappa}(M) \to c_{X_0}(\pi(M)) = K_1$ belongs to $\mathcal{E}$, which is the same as to say that π is c-preserving, and for that we proceed inductively. Since $v^1_1 = 1$, there is nothing to be shown in case $\kappa = 1$. If $\kappa = \alpha + 1$ is a successor, then $v^\kappa_1 = v^\alpha_1 \cdot v^\kappa_\alpha$ belongs to $\mathcal{E}$, since $v^\alpha_1 \in \mathcal{E}$ by hypothesis and since $v^\kappa_\alpha \in \mathcal{E}$ because of the c-preservation of

$$p^\kappa_\alpha \cong (X_\alpha \times X^\alpha \to X^\alpha).$$

Hence we are left with having to deal with the case of a limit ordinal κ. We prove that every $y_1 : P \to K_1$ with $P \in \mathcal{P}$ factors through v^κ_1. For that one constructs inductively morphisms y_γ with $v^\gamma_\alpha \cdot y_\gamma = y_\alpha$ $(1 \leq \alpha \leq \gamma \leq \kappa)$, as follows. If $\gamma = \alpha + 1$ is a successor ordinal, for the existing y_α one obtains y_γ with $v^\gamma_\alpha \cdot y_\gamma = y_\alpha$ since p^γ_α is c-preserving, so that v^γ_α belongs to $\mathcal{E}$. If γ is a limit ordinal, then the existing morphisms y_α $(1 \leq \alpha < \gamma)$ induce a morphism $x : P \to X^\gamma$ with

$$q^\gamma_\beta \cdot x = q^\alpha_\beta \cdot k_\alpha \cdot y_\alpha\ (0 \leq \beta < \gamma,\ \alpha = \beta + 1),$$

because of the product property of X^γ. It now suffices to show that x factors through the monomorphism $k_\gamma := c_{X^\gamma}(p^\kappa_\gamma(m)) : K_\gamma \to X^\gamma$, i.e., $\widehat{x} := x(1_P) \leq k_\gamma$. But since c satisfies FSPP, one may just check whether

$$p^\gamma_F(\widehat{x}) \leq c_{X^F}(p^\gamma_F(p^\kappa_\gamma(m))) = c_{X^F}(p^\kappa_F(m))$$

holds for every finite set $F \subseteq \gamma = \{\beta : 0 \leq \beta < \gamma\}$. Choose $\alpha < \gamma$ such that $F \subseteq \alpha$. Since obviously $p^\gamma_\alpha \cdot x = k_\alpha \cdot y_\alpha$, one has

$$p^\gamma_F(\widehat{x}) = p^\alpha_F(p^\gamma_\alpha(\widehat{x})) \leq p^\alpha_F(c_{X^\alpha}(p^\kappa_\alpha(m))) \leq c_{X^F}(p^\kappa_F(m)).$$

$\square$

4. Examples

1. The category $\mathcal{T}op$ of topological spaces with its natural (surjective, embedding)-factorization system and its natural closure operator obviously satisfies all needed assumptions. The Theorem therefore yields the classical Tychonoff Theorem [11] for compact spaces which, according to Kuratowski [9] and Mrówka [10], fit into the definition of compactness used here.

2. Čech's [2] product theorem for compact pretopological spaces also follows from the Theorem. The proof that compactness is characterized by closedness of product projections was given by Dikranjan and Giuli [4].

3. A continuous map $f : X \to Y$ of topological spaces is *proper* if it is compact as an object in the category of spaces over Y which inherits the natural factorization structure and the natural closure operator from $\mathcal{T}op$; hence proper maps are stably-closed maps as in Bourbaki [1] (also known as perfect maps when X is Hausdorff, cf. [6]). The Theorem shows that the fibred product of proper maps is proper. From this fact one derives immediately Frolík's [8] generalization of Tychonoff's Theorem, namely that the direct product of proper maps is proper.

4. For the category of topological groups and its usual closure, it is not known whether categorically compact objects (in the sense of 2) are compact in the usual topological sense. Recently Dikranjan and Uspenskij have given a filter-theoretic proof for a product theorem for categorically compact topological groups (private communication), but the result also follows immediately from the Theorem proved here.

5. We are not able to apply the Theorem to the category $\mathcal{L}oc$ of locales and its "slices" (cf. [12]) since we are not aware of a suitable factorization system $(\mathcal{E}, \mathcal{M})$ in $\mathcal{L}oc$ with a surjectivity class $\mathcal{E}$ as defined in 2. Note, however, that the Tychonoff Theorem holds true in $\mathcal{L}oc$, even without the assumption of a choice principle. On the other hand, in our Theorem we cannot dispense of the Axiom of Choice since the classical result in $\mathcal{T}op$ requires it by necessity.

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