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ON THE SET OF ALL CONTINUOUS FUNCTIONS WITH UNIFORMLY CONVERGENT FOURIER SERIES

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ABSTRACT. In this article we calculate the exact location in the Borel hierarchy of UCF, the set of all continuous functions on the unit circle with uniformly convergent Fourier series. It turns out to be complete $F_{\sigma\delta}$. Also we prove that any $G_{\delta\sigma}$ set that includes UCF must contain a continuous function with divergent Fourier series.

INTRODUCTION

There are many criteria for uniform convergence of a Fourier series on the unit circle. One can find those tests in [Zy]. In the present paper, we study UCF from the point of view of descriptive set theory. In [Ke], it was a conjecture that UCF is complete $F_{\sigma\delta}$ (i.e., $F_{\sigma\delta}$ but not $G_{\delta\sigma}$). A lot of natural complete $F_{\sigma\delta}$ sets have been found. For example, the collection of reals that are normal or simply normal to base n [KL]; $C^\infty(\mathbb{T})$, the class of infinitely differentiable functions (viewed as a 2π-periodic function on $\mathbb{R}$); and UC_X, the class of convergent sequences in a separable Banach space X, are complete $F_{\sigma\delta}$ [Ke]. It turns out that UCF is complete $F_{\sigma\delta}$. We give two different proofs for it. Ajtai and Kechris [AK] have shown that ECF, the set of all continuous functions with everywhere convergent Fourier series, is complete CA, i.e., coanalytic non-Borel. We show that there is no $G_{\delta\sigma}$ set A such that $UCF \subseteq A \subseteq ECF$. Hence any $G_{\delta\sigma}$ set that includes UCF must contain a continuous function with divergent Fourier series. From this point of view, although there are many natural complete $F_{\sigma\delta}$ sets, we can claim that UCF is a very interesting set in analysis.

DEFINITIONS AND BACKGROUND

Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ be the set of positive integers and $\mathbb{N}^{\mathbb{N}}$ the Polish space with the usual product topology and $\mathbb{N}$ discrete. Let X be a Polish space. A subset A of X is CA if there is a Borel funtion from $\mathbb{N}^{\mathbb{N}}$ to X such that $f\left(\mathbb{N}^{\mathbb{N}}\right) = X - A$. A CA ($F_{\sigma\delta}$) subset A of X is called complete CA ($F_{\sigma\delta}$) if for any CA ($F_{\sigma\delta}$) subset B of $\mathbb{N}^{\mathbb{N}}$, there is a Borel (continuous) function f from $\mathbb{N}^{\mathbb{N}}$ to X such that the preimage of A of f is B, i.e., $B = f^{-1}(A)$. From the definition, it is easy to see that no

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complete CA ($F_{\sigma\delta}$) set is Borel ($G_{\delta\sigma}$). In paticular, if an $F_{\sigma\delta}$ subset A of a Polish space is complete $F_{\sigma\delta}$ and the continuous preimage of an $F_{\sigma\delta}$ subset B of a Polish space, then B is also complete $F_{\sigma\delta}$.

Let $\mathbb{R}$ be the set of real numbers. Let $\mathbb{T}$ denote the unit circle and I the unit interval. Let E be $\mathbb{T}$ or I. We denote by $C(E)$ the Polish space of continuous functions on E with the uniform metric

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in E\}.$$

$C(\mathbb{T})$ can also be considered as the space of all continuous 2π-periodic functions on $\mathbb{R}$, viewing $\mathbb{T}$ as $\mathbb{R}/2\pi\mathbb{Z}$. Let UC denote the set of all sequences of continuous functions on I that are uniformly convergent, i.e.,

$$UC = \{(f_n) \in C(I)^{\mathbb{N}} : (f_n) \text{ converges uniformly } \}.$$

To each $f \in C(\mathbb{T})$, we associate its Fourier series

$$S[f] \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx},$$

where $\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt$. Let

$$S_n(f,t) = \sum_{k=-n}^{n} \hat{f}(k) e^{ikt}$$

be the nth partial sum of the Fourier series of f. We say the Fourier series of f converges at a point $t \in \mathbb{T}$ if the sequence $(S_n(f,t))$ converges. Similarly, we define the uniform convergence of the Fourier series of f. Let ECF denote the set of all continuous functions with everywhere convergent Fourier series. According to a standard theorem [Ka], the Fourier series of f at t converges to $f(t)$ if it converges. Hence we have

$$\begin{aligned} ECF =& \{f \in C(\mathbb{T}) : \forall\, t \in [0,2\pi] \ \Big((S_n(f,t)) \text{ converges } \Big) \} \\ =& \{f \in C(\mathbb{T}) : \forall\, t \in [0,2\pi] \ \Big(f(t) = \lim_{n\to\infty} S_n(f,t) \Big) \}. \end{aligned}$$

We denote by NCF the complement of ECF. Let UCF denote the set all continuous functions with uniformly convergent Fourier series, i.e.,

$$UCF = \{f \in C(\mathbb{T}) : \text{ the Fourier series of } f \text{ converges uniformly } \}.$$

Results

Theorem ([AK])**.** *ECF is complete CA.*

(See [AK].)

Proposition 1. *UCF and UC are $F_{\sigma\delta}$.*

Proof. Let $\mathbb{Q}$ be the set of all rational numbers. We consider $\mathbb{T}$ as $[0, 2\pi]$ with 0 and 2π identified. By the definition of UCF,

$$\begin{aligned} f \in UCF &\iff S_N(f) \text{ converges uniformly} \\ &\iff \forall\, a \in \mathbb{N}\ \exists\, b \in \mathbb{N}\ \forall\, c, d \in \mathbb{N}\ \forall\, e \in \mathbb{Q}\Big(|S_{b+c}(f, e) - S_{b+d}(f, e)| \leq \frac{1}{a}\Big) \\ &\iff f \in \bigcap_{a\in\mathbb{N}} \bigcup_{b\in\mathbb{N}} \bigcap_{c,d\in\mathbb{N}} \bigcap_{e\in\mathbb{Q}\cap[0,2\pi]} V(a, b, c, d, e), \end{aligned}$$

where $V(a, b, c, d, e)$ is the collection of $f \in C(\mathbb{T})$ such that $|S_{b+c}(f, e)-S_{b+d}(f, e)| \leq 1/a$, which is closed, since the function $f \longmapsto \hat{f}(n)$ is continuous. Hence UFC is $F_{\sigma\delta}$. Similarly, so is UC. We are done. □

Lemma 2. *The set $C_3 = \{\alpha \in \mathbb{N}^{\mathbb{N}} : \lim_{n\to\infty} \alpha(n) = \infty\}$ is complete $F_{\sigma\delta}$.*

(See [Ke, p. 180].) This set will be used to prove our main theorem.

Proposition 3. *UC is complete $F_{\sigma\delta}$.*

Proof. We define the function F from $\mathbb{N}^{\mathbb{N}}$ to $C(I)^{\mathbb{N}}$ as follows: for each $\beta \in \mathbb{N}^{\mathbb{N}}$,

$$F(\beta) = \Big(\frac{1}{\beta(n)}\Big).$$

Then it is easy to see that

$$\beta \in C_3 \iff F(\beta) \text{ converges to } 0 \iff F(\beta) \text{ converges uniformly},$$

since $F(\beta)$ is a sequence of constant functions. Clearly, F is continuous. Hence UC is the continuous preimage of C_3. By Proposition 1 and Lemma 2, UC is complete $F_{\sigma\delta}$. □

Theorem 4. *There is a continuous function H from $\mathbb{N}^{\mathbb{N}}$ to $C(\mathbb{T})$ such that for all A with $UCF \subseteq A \subseteq ECF$,*

$$\beta \in C_3 \iff H(\beta) \in A,$$

and

$$\beta \notin C_3 \iff H(\beta) \in NCF.$$

In particular, UCF is complete $F_{\sigma\delta}$.

By this theorem, we have the following corollary.

Corollary 5. *There is no $G_{\delta\sigma}$ set A such that*

$$UCF \subseteq A \subseteq ECF,$$

i.e., any $G_{\delta\sigma}$ set that includes UCF must contain a continuous function with divergent Fourier series.

Proof. Suppose a $G_{\delta\sigma}$ set A satisfies $UCF \subseteq A \subseteq ECF$. Then by Theorem 4, we obtain $H^{-1}(A) = C_3$. Since A is $G_{\delta\sigma}$, so is C_3. By Lemma 2, this contradicts our assumption. □

It is a basic fact of descriptive set theory [Ke] that any Borel set is coanalytic. Since ECF is complete CA by Theorem [AK], it is a very natural guess that the complement of C_3 can be reducible to $ECF - UCF$. In fact, we have the following theorem.

Theorem 6. *There is a continuous function $\widetilde{H}$ from $\mathbb{N}^{\mathbb{N}}$ to $C(\mathbb{T})$ such that*

$$\beta \in C_3 \iff \widetilde{H}(\beta) \in UCF,$$

and

$$\beta \notin C_3 \iff \widetilde{H}(\beta) \in ECF - UCF.$$

In particular, UCF is complete $F_{\sigma\delta}$.

In order to prove Theorem 4 and Theorem 6, we need the following criterion due to Dini and Lipschitz [Zy, p. 63]. Let f be defined in a closed interval J, and let

$$\omega(\delta) = \omega(\delta; f) = \sup\{|f(x) - f(y)| : x, y \in J \text{ and } |x - y| \leq \delta\}.$$

The function $\omega(\delta)$ is called the *modulus of continuity* of f.

The Dini-Lipschitz test. *If f is continuous and its modulus of continuity $\omega(\delta)$ satisfies the condition $\omega(\delta)\log\delta \to 0$, then the Fourier series of f converges uniformly.*

We introduce the Féjer polynomials, for given $0 < n < N \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$Q(x, N, n) = 2\sin Nx \sum_{k=1}^{n} \frac{\sin kx}{k},$$
$$R(x, N, n) = 2\cos Nx \sum_{k=1}^{n} \frac{\sin kx}{k}.$$

These two polynomials were used in [Zy] to prove that there exists a continuous function whose Fourier series diverges at a point.

Lemma 7. *There are positive numbers $C_1, C_2 > 0$ such that*

$$|Q| < C_1 \text{ and } |R| < C_2,$$

i.e., these polynomials are uniformly bounded in x, N, n.

Since

$$\sum_{k=1}^{n} \frac{\sin kx}{k}$$

is uniformly bounded in n and x, Lemma 7 follows. By Lemma 7, we immediately have the following.

Proposition 8. *Let (N_k) and (n_k) be any two sequences of positive integers, with $n_k < N_k$, and let (α_k) be a sequence of real numbers such that $\alpha_1 + \alpha_2 + \alpha_3 + \cdots < \infty$. Then the series $\sum \alpha_k Q(x, N_k, n_k)$, $\sum \alpha_k R(x, N_k, n_k)$ converge to continuous functions.*

Proof of Theorem 4. We fix A with $UCF \subseteq A \subseteq ECF$. Let $\alpha_k = 2^{-k}$, $n_k = N_k/2 = 2^{2^k}$ $(k = 1, 2, 3, \cdots)$. We define H from $\mathbb{N}^{\mathbb{N}}$ to $C(\mathbb{T})$ as follows: for all $\beta \in \mathbb{N}^{\mathbb{N}}$,

$$H(\beta) = \sum \alpha_k \frac{1}{\beta(k)} Q(x, N_k, n_k).$$

Claim 1. *H is continuous and well-defined.*

Proof. By Proposition 8, H is well-defined. By Lemma 7, it is easy to see that H is continuous.

We divide the rest of proof into two parts so that we have more intuition.

Case 1. $\lim_{n\to\infty} \beta(k) \neq \infty$.

We want to show that $H(\beta) \in NCF$. For each $k \in \mathbb{N}$ the inequality

$$\begin{aligned} &\left|S_{N_k+n_k}\big(H(\beta), 0\big) - S_{N_k}\big(H(\beta), 0\big)\right| = \left|\sum_{|l|\le N_k+n_k} \widehat{H(\beta)}(l) - \sum_{|l|\le N_k} \widehat{H(\beta)}(l)\right| \\ &= \alpha_k \frac{1}{\beta(k)}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n_k}\right) > \alpha_k \frac{1}{\beta(k)} \log n_k = 2^{-k} \frac{1}{\beta(k)} \log 2^{2^k} \\ &= \frac{1}{\beta(k)} \log 2 \end{aligned} \tag{1}$$

holds. Since $\lim_{n\to\infty} \beta(k) \neq \infty$, there exists a $p \in \mathbb{N}$ such that for infinitely many k's, $\beta(k) = p$. Hence the Fourier series of $H(\beta)$ does not converge, since in (1) we have $1/p \log 2$ for infinitely many k's. So we derive $H(\beta) \in NCF$.

Case 2. $\lim_{n\to\infty} \beta(k) = \infty$.

We show that $H(\beta) \in UCF$. We will demonstrate that $\omega(\delta; H(\beta)) \log \delta \to 0$ as $\delta \to 0$. Then by the Dini-Lipschitz test, this shows that the Fourier series of $H(\beta)$ converges uniformly. We take any $0 < \delta \le 1/2$ and define $\nu = \nu(\delta)$ as the largest integer k satifying $2^{2^k} \le 1/\delta$ By Lemma 7, we have the following inequality:

$$\begin{aligned} &\left|\sum_{k=\nu+1}^{\infty} \alpha_k \frac{1}{\beta(k)} Q(x+\delta, N_k, n_k) - \sum_{k=\nu+1}^{\infty} \alpha_k \frac{1}{\beta(k)} Q(x, N_k, n_k)\right| \\ &\le 2C \sum_{k=\nu+1}^{\infty} \alpha_k \frac{1}{\beta(k)} \le 2C \sup\{\frac{1}{\beta(k)} : k > \nu\} \sum_{k=\nu+1}^{\infty} \alpha_k \\ &= 4C \sup\{\frac{1}{\beta(k)} : k > \nu\} 2^{-\nu-1} \le 4C \sup\{\frac{1}{\beta(k)} : k > \nu\} \frac{\log 2}{|\log \delta|}. \end{aligned} \tag{2}$$

Now we calculate the rest of $H(\beta)$. We clearly have

$$Q'(x, N, n) = NR(x, N, n) + 2 \sin Nx \sum_{k=1}^{n} \cos kx, \ |Q'| \le NC + 2n = nC_1,$$

for $N = 2n$ and $C_1 = 2C + 2$. By the mean value theorem, we have the following inequality:

$$\begin{aligned}
&\left|\sum_{k\le\nu}\alpha_k\frac{1}{\beta(k)}Q(x+\delta,N_k,n_k)-\sum_{k\le\nu}\alpha_k\frac{1}{\beta(k)}Q(x,N_k,n_k)\right|\\
&\le C_1\delta\left(2^{-1}2^{2^1}\frac{1}{\beta(1)}+\cdots+2^{-\nu}2^{2^\nu}\frac{1}{\beta(\nu)}\right)\\
&\le C_1 2^{2^{-\nu}}\sum_{k\le\nu}2^{-k}2^{2^k}\frac{1}{\beta(k)}\le C_1\frac{1}{|\log\delta|}2^{2^{-\nu}}\sum_{k\le\nu}2^{2^k-k}\frac{1}{\beta(k)}.
\end{aligned} \tag{3}$$

By (2) and (3), we have the following:

$$\left|\omega(\delta;H(\beta))\log\delta\right| \le \max\{4C\sup\{\frac{1}{\beta(k)}:k>\nu\}\log 2, C_1 2^{2^{-\nu}}\sum_{k\le\nu}2^{2^k-k}\frac{1}{\beta(k)}\}. \tag{4}$$

Now if $\delta \to 0$, then $\nu \to \infty$. So it suffices to show that the right part of (4) goes to 0 as $\nu \to \infty$. Since $\beta(\nu) \to \infty$ as $\nu \to \infty$, $\sup\{1/\beta(k) : k > \nu\}$ goes to 0. We need to show that the rest goes to zero as ν diverges to infinity. This requires the following small claim.

Claim 2. $\sum_{k\le\nu} 2^{2^k-k} \le 2^{2^\nu-\nu+1}$.

Proof. Use induction on ν. For $\nu = 1$, $2^{2-1} = 2 \le 2^{2-1+1} = 2^2$. Suppose it is true for ν. By the induction assumption, $\sum_{k\le\nu} 2^{2^k-k} + 2^{2^{\nu+1}-(\nu+1)} \le 2^{2^\nu-\nu+1} + 2^{2^{\nu+1}-(\nu+1)}$. It is enough to show that $2^{2^\nu-\nu+1} + 2^{2^{\nu+1}-(\nu+1)} \le 2^{2^{\nu+1}+1}$. Letting $\theta = 2^{2^\nu}$, one can verify this inequality.

Fix ϵ. Take N_0 such that $1/\beta(k) < \epsilon$ for all $k \ge N_0$. For this N_0, we choose $N > N_0$ so that $2^{-2^\nu+\nu}\sum_{k\le N_0} 2^{2^k-k} < \epsilon$ for all $\nu \ge N$. Then for all $\nu \ge N$, by claim 2, the following inequality is valid:

$$\begin{aligned}
&2C_1 2^{-2^\nu}2^\nu\sum_{k\le\nu}2^{2^k-k}\\
&< 2C_1\left(2^{-2^\nu+\nu}\sum_{k\le N_0}2^{2^k-k}\frac{1}{\beta(k)}+2^{-2^\nu+\nu}\sum_{N_0<k\le\nu}2^{2^k-k}\frac{1}{\beta(k)}\right)\\
&< 2C_1\left(\epsilon+2^{-2^\nu+\nu}\epsilon\sum_{N_0<k\le\nu}2^{2^k-k}\right)<2\epsilon C_1\left(1+\frac{2^{2^\nu-\nu+4}}{2^{2^\nu-\nu}}\right)\\
&= 34\epsilon C_1.
\end{aligned}$$

Hence the right side of (4) converges to zero as ν goes to the infinity, i.e., as $\delta \to 0$. So we derive $H(\beta) \in UCF$.

By case 1 and case 2, we obtain

$$\beta \notin C_3 \Rightarrow\ H(\beta) \in NCF \text{ and}$$
$$\beta \in C_3 \Rightarrow\ H(\beta) \in UCF,$$

respectively. Since NCF and A are disjoint, we have the following:

$$\beta \notin C_3 \iff H(\beta) \in NCF \text{ and}$$
$$\beta \in C_3 \iff H(\beta) \in A.$$

We have shown the first assertion of the theorem. In particular, C_3 is the preimage of UCF. Hence by Lemma 2, the second assertion follows. We have completed the proof of Theorem 4. □

Proof of Theorem 6. Instead of Q, we use R. With N_k, n_k, and α_k as in the proof of Theorem 4, we define $\widetilde{H}$ from $\mathbb{N}^{\mathbb{N}}$ to $C(\mathbb{T})$ as follows: for each $\beta \in \mathbb{N}^{\mathbb{N}}$,

$$\widetilde{H}(\beta) = \sum \alpha_k \frac{1}{\beta(k)} R(x, N_k, n_k).$$

The same proof as before will demonstrate that this function is well-defined and continuous and that if $\lim_{n\to\infty} \beta(n) = \infty$, then the Fourier series of $\widetilde{H}(\beta)$ converges uniformly. So it suffices to show that if $\lim_{n\to\infty} \beta(n) \neq \infty$, then $\widetilde{H}(\beta) \in ECF - UCF$. Suppose $\lim_{n\to\infty} \beta(n) \neq \infty$. The representation of $\widetilde{H}(\beta)$ as Fourier series is $\sum a_v \sin vx$. We see that $\sum a_v \sin vx$ converges uniformly for $\delta \leq |x| \leq \pi$ for any $\delta > 0$, since the partial sums of $R(x, N_k, n_k)$ are uniformly bounded in k and x, for $\delta \leq |x| \leq \pi$. The series $\sum a_v \sin vx$ contains sines only, and hence it converges for $x = 0$, and so everywhere. Now we will show that $\sum a_v \sin vx$ does not converge uniformly. It is easy to see that

$$\sum_{v=1}^{3n_k} a_v \sin vx - \sum_{v=1}^{2n_k} a_v \sin vx = \sum_{v=2n_k+1}^{3n_k} a_v \sin vx = 2^{-k} \frac{1}{\beta(k)} \sum_{v=1}^{n_k} \frac{\sin(2n_k + v)x}{v}.$$

So if we let $x = \pi/4n_k$, then we have

$$\left|\sum_{v=1}^{3n_k} a_v \sin vx - \sum_{v=1}^{2n_k} a_v \sin vx\right| = 2^{-k} \frac{1}{\beta(k)} \sum_{v=1}^{n_k} \frac{\sin(2n_k + v)x}{v} \geq 2^{-k} \frac{1}{\beta(k)} \sin\frac{\pi}{4} \sum_{v=1}^{n_k} \frac{1}{v},$$

since, for all v in the interval $1 \leq v \leq n_k$,

$$\frac{3}{4}\pi \geq \frac{\pi}{4n_k}(2n_k + v) \geq \frac{\pi}{2}.$$

So finally,

$$\left|\sum_{v=1}^{3n_k} a_v \sin vx - \sum_{v=1}^{2n_k} a_v \sin vx\right| \geq 2^{-k} \frac{1}{\beta(k)} \sin\frac{\pi}{4} \sum_{v=1}^{n_k} \frac{1}{v} \geq 2^{-k} \frac{1}{\beta(k)} \log n_k \sin\frac{\pi}{4} = \frac{\log 2}{\sqrt{2}} \frac{1}{\beta(k)}. \tag{5}$$

Hence $\sum a_v \sin vx$ does not converge uniformly, since in (5) the same value appears for infinitely many k's. Hence, as in the proof of Theorem 4, we finish the proof of Theorem 6. □

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