

## COLORING FINITE SUBSETS OF UNCOUNTABLE SETS

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ABSTRACT. It is consistent for every  $1 \leq n < \omega$  that  $2^\omega = \omega_n$  and there is a function  $F : [\omega_n]^{<\omega} \rightarrow \omega$  such that every finite set can be written in at most  $2^n - 1$  ways as the union of two distinct monocolored sets. If GCH holds, for every such coloring there is a finite set that can be written at least  $\frac{1}{2} \sum_{i=1}^n \binom{n+i}{n} \binom{n}{i}$  ways as the union of two sets with the same color.

### 1. INTRODUCTION

In [6] we proved that for every coloring  $F : [\omega_n]^{<\omega} \rightarrow \omega$  there exists a set  $A \in [\omega_n]^{<\omega}$  which can be written at least  $2^n - 1$  ways as  $A = H_0 \cup H_1$  for some  $H_0 \neq H_1$ ,  $F(H_0) = F(H_1)$ , and that for  $n = 1$  there is in fact a function  $F$  for which this is sharp. Here we show that for every  $n < \omega$  it is consistent that  $2^\omega = \omega_n$  and for some function  $F$  as above for every finite set  $A$  there are at most  $2^n - 1$  solutions of the above equation. We use historic forcing, which was first used in [1] and [7], then in [5] and [4]. Under GCH, we improve the positive result of [6] by showing that for every  $F$  as above some finite set can be written at least  $T_n = \frac{1}{2} \sum_{i=1}^n \binom{n+i}{n} \binom{n}{i}$  ways as the union of two sets with the same  $F$  value.

With the methods of [6] it is easy to show the following corollary of our independence result. It is consistent that  $2^\omega = \omega_n$  and there is a function  $f : \mathbb{R} \rightarrow \omega$  such that if  $x$  is a real number then  $x$  cannot be written more than  $2^n - 1$  ways as the arithmetic mean of some  $y \neq z$  with  $f(y) = f(z)$ . ( $(y, z)$  and  $(z, y)$  are not regarded as distinct.)

**Notation.** We use the standard set theory notation. If  $S$  is a set,  $\kappa$  a cardinal, then  $[S]^\kappa = \{A \subseteq S : |A| = \kappa\}$ ,  $[S]^{<\kappa} = \{A \subseteq S : |A| < \kappa\}$ ,  $[S]^{\leq \kappa} = \{A \subseteq S : |A| \leq \kappa\}$ .  $P(S)$  is the power set of  $S$ . If  $f$  is a function,  $A$  a set, then  $f[A] = \{f(x) : x \in A\}$ .

### 2. THE INDEPENDENCE RESULT

**Theorem 1.** *For  $1 \leq n < \omega$  it is consistent that  $2^\omega = \omega_n$  and there is a function  $F : [\omega_n]^{<\omega} \rightarrow \omega$  such that for every  $A \in [\omega_n]^{<\omega}$  there are at most  $2^n - 1$  solutions of  $A = H_0 \cup H_1$  with  $H_0 \neq H_1$ ,  $F(H_0) = F(H_1)$ .*

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For  $\alpha < \omega_n$  fix a bijection  $\varphi_\alpha : \alpha \rightarrow |\alpha|$ . For  $x \in [\omega_n]^{<\omega}$  define  $\gamma_i(x)$  for  $i < k = \min(n, |x|)$  as follows:  $\gamma_0(x) = \max(x)$ ,

$$\gamma_{i+1}(x) = \varphi_{\gamma_0(x)}^{-1} \left( \gamma_i(\varphi_{\gamma_0(x)}[x \cap \gamma_0(x)]) \right),$$

and  $\gamma(x) = \{\gamma_0(x), \dots, \gamma_{k-1}(x)\}$ .

So, for example, if  $n = 0$  then  $\gamma(x) = \emptyset$ , if  $n = 1$ ,  $x \neq \emptyset$ , then  $\gamma(x) = \{\gamma_0(x)\} = \{\max(x)\}$ .

**Lemma 1.** *Given  $s \in [\omega_n]^{\leq n}$ , there are at most countably many  $x \in [\omega_n]^{<\omega}$  such that  $\gamma(x) = s$ .*

*Proof.* By induction on  $n$ . □

Let  $\Phi(s) = \bigcup \{x : \gamma(x) \subseteq s\}$ , a countable set for  $s \in [\omega_n]^{<\omega}$ .

**Definition.** The two sets  $x, y \in [\omega_n]^{<\omega}$  are *isomorphic* if the structures  $(x; <, \gamma_0(x), \dots, \gamma_{k-1}(x))$ ,  $(y; <, \gamma_0(y), \dots, \gamma_{k-1}(y))$  are isomorphic, i.e.,  $|x| = |y|$  and the positions of the elements  $\gamma_i(x)$ ,  $\gamma_i(y)$  are the same.

Notice that for every finite  $j$  there are just finitely many isomorphism types of  $j$ -element sets.

The elements of  $P$ , the applied notion of forcing, will be *some* structures of the form  $p = (s, f)$  where  $s \in [\omega_n]^{<\omega}$  and  $f : P(s) \rightarrow \omega$ .

The only element of  $P_0$  is  $\mathbf{1}_P = (\emptyset, \langle \emptyset, 0 \rangle)$ ; it will be the largest element of  $P$ . The elements of  $P_1$  are of the form  $p = (\{\xi\}, f)$  where  $f(\emptyset) = 0 \neq f(\{\xi\})$  for  $\xi < \omega_n$ .

Given  $P_t$ ,  $p = (s, f)$  is in  $P_{t+1}$  if the following is true.  $s = \Delta \cup a \cup b$  is a disjoint decomposition.  $p' = (\Delta \cup a, f')$  and  $p'' = (\Delta \cup b, f'')$  are in  $P_t$ , where  $f' = f|P(\Delta \cup a)$ ,  $f'' = f|P(\Delta \cup b)$ . There is  $\pi : \Delta \cup a \rightarrow \Delta \cup b$ , an isomorphism between  $(\Delta \cup a, <, P(\Delta \cup a), f')$  and  $(\Delta \cup b, <, P(\Delta \cup b), f'')$ .  $\pi|_\Delta$  is the identity. For  $H \subseteq \Delta \cup a$  the sets  $H$  and  $\pi[H]$  are isomorphic.  $a \cap \Phi(\Delta) = b \cap \Phi(\Delta) = \emptyset$ .  $f - f' - f''$  is one-to-one and takes only values outside  $\text{Ran}(f')$  (which is the same as  $\text{Ran}(f'')$ ).  $P = \bigcup \{P_t : t < \omega\}$ . We make  $p \leq p', p''$  and the ordering on  $P$  is the one generated by this.

**Lemma 2.**  $(P, \leq)$  is ccc.

*Proof.* Assume that  $p_\alpha \in P$  ( $\alpha < \omega_1$ ). We can assume by thinning and using the  $\Delta$ -system lemma and the pigeonhole principle that the following hold.  $p_\alpha \in P_t$  for the same  $t < \omega$ .  $p_\alpha = (\Delta \cup a_\alpha, f_\alpha)$  where the structures  $(\Delta \cup a_\alpha, <, P(\Delta \cup a_\alpha), f_\alpha)$  and  $(\Delta \cup a_\beta, <, P(\Delta \cup a_\beta), f_\beta)$  are isomorphic for  $\alpha, \beta < \omega_1$ ,  $\{\Delta, a_\alpha : \alpha < \omega_1\}$  pairwise disjoint. We can also assume that if  $\pi$  is the isomorphism between  $(\Delta \cup a_\alpha, <, f_\alpha)$  and  $(\Delta \cup a_\beta, <, f_\beta)$  then  $H$  and  $\pi[H]$  are isomorphic for  $H \subseteq \Delta \cup a_\alpha$ . Moreover, if we assume that  $\Delta$  occupies the same positions in the ordered sets  $\Delta \cup a_\alpha$  ( $\alpha < \omega_1$ ) then  $\pi$  will be the identity on  $\Delta$ . As  $\Phi(\Delta)$  is countable, by removing countably many indices we can also assume that  $\Phi(\Delta) \cap a_\alpha = \emptyset$  for  $\alpha < \omega_1$ . Now any  $p_\alpha$  and  $p_\beta$  are compatible, as we can take  $p = (\Delta \cup a_\alpha \cup a_\beta, f) \leq p_\alpha, p_\beta$  where  $f \supseteq f_\alpha, f_\beta$  is an appropriate extension, i.e.,  $f - f_\alpha - f_\beta$  is one-to-one and takes values outside  $\text{Ran}(f_\alpha)$ . □

**Lemma 3.** *If  $(s, f) \in P$  and  $H_0, H_1 \subseteq s$  have  $f(H_0) = f(H_1)$ , then  $H_0, H_1$  are isomorphic.*

*Proof.* Set  $(s, f) \in P_t$ . We prove the statement by induction on  $t$ . There is nothing to prove for  $t < 2$ . Assume now that  $(s, f) \in P_{t+1}$ ,  $s = \Delta \cup a \cup b$ ,  $\pi : \Delta \cup a \rightarrow \Delta \cup b$  as in the definition of  $(P, \leq)$ . As  $f(H_0)$  is a value taken twice by  $f$ , both  $H_0$  and  $H_1$  must be subsets of either  $\Delta \cup a$  or  $\Delta \cup b$ . We are done by induction unless  $H_0 \subseteq \Delta \cup a$  and  $H_1 \subseteq \Delta \cup b$  (or vice versa). Now  $H_0$  and  $\pi[H_0]$  are isomorphic and  $f(H_0) = f(\pi[H_0]) = f(H_1)$ , so by the inductive hypothesis  $\pi[H_0]$  and  $H_1$  are isomorphic, and then so are  $H_0, H_1$ .  $\square$

**Lemma 4.** *If  $(s, f) \in P$ ,  $H_0, H_1 \subseteq s$ ,  $f(H_0) = f(H_1)$ ,  $x \in H_0 \cap H_1$ , then  $x$  occupies the same position in the ordered sets  $H_0, H_1$ .*

*Proof.* Similarly to the proof of the previous lemma, by induction on  $t$ , for  $(s, f) \in P_t$ . With similar steps, we can assume that  $(s, f) = (\Delta \cup a \cup b, f) \leq (\Delta \cup a, f'), (\Delta \cup b, f'')$ ,  $H_0 \subseteq \Delta \cup a$ ,  $H_1 \subseteq \Delta \cup b$ . Notice that  $x \in \Delta$ . Now, as  $\pi(x) = x$ ,  $x$  is a common element of  $\pi[H_0]$  and  $H_1$ , and also  $f''(\pi[H_0]) = f''(H_1)$ . By induction we get that  $x$  occupies the same position in  $\pi[H_0]$  and  $H_1$ , so by pulling back we get that this is true for  $H_0$  and  $H_1$ .  $\square$

**Lemma 5.** *If  $(s, f) \in P$ ,  $A \subseteq s$ ,  $0 \leq j \leq n$ , then  $A$  can be written at most  $2^j - 1$  ways as  $A = H_0 \cup H_1$  with  $H_0, H_1$  distinct,  $f(H_0) = f(H_1)$ , and  $|\gamma(H_0) \cap \gamma(H_1)| \geq n - j$ .*

*Proof.* By induction on  $j$  and, inside that induction, by induction on  $t$ , for  $(s, f) \in P_t$ . The case  $t < 2$  will always be trivial.

Assume first that  $j = 0$ . In this case our lemma reduces to the following statement. There are no  $H_0 \neq H_1$  such that  $f(H_0) = f(H_1)$  and  $\gamma(H_0) = \gamma(H_1)$ . In the inductive argument we assume as usual that  $s = \Delta \cup a \cup b$  and so  $(s, f) \in P_{t+1}$  was created from  $(\Delta \cup a, f')$  and  $(\Delta \cup b, f'')$ ,  $H_0 \subseteq \Delta \cup a$ ,  $H_1 \subseteq \Delta \cup b$ . As  $\gamma(H_0) = \gamma(H_1)$ ,  $\gamma(H_0) \subseteq \Delta$ , but then, as  $\Phi(\Delta) \cap a = \emptyset$ ,  $H_0$  can have no points outside  $\Delta$  and similarly for  $H_1$ , so we can go back, say to  $(\Delta \cup a, f') \in P_t$ , which concludes the argument.

Assume now that the statement is proved for  $j$  and we have  $p = (s, f) \in P_{t+1}$ ,  $s = \Delta \cup a \cup b$  and  $p$  was created from  $p' = (\Delta \cup a, f')$  and  $p'' = (\Delta \cup b, f'')$ . In  $A \subseteq \Delta \cup a \cup b$  we can assume that  $y = A \cap a \neq \emptyset$ ,  $z = A \cap b \neq \emptyset$ , as otherwise we can pull back to  $p'$  or  $p''$ . But then, if  $A = H_0 \cup H_1$ , then, if, say,  $H_0 \subseteq \Delta \cup a$ ,  $H_1 \subseteq \Delta \cup b$  hold, then necessarily  $H_0 \cap a = y$ ,  $H_1 \cap b = z$ , so  $H_0 = x_0 \cup y$ ,  $H_1 = x_1 \cup z$  where  $x_0 \cup x_1 = x = A \cap \Delta$ . We can create decompositions of  $B = x \cup \pi[y] \cup z$  by taking  $B = \pi[H_0] \cup H_1$ . But some of these decompositions will not be different and it may happen that we get a non-proper (i.e., one-piece) decomposition. This can only happen if  $\pi[y] = z$ , and then the two decompositions  $A = (x_0 \cup y) \cup (x_1 \cup z)$  and  $A = (x_1 \cup y) \cup (x_0 \cup y)$  produce the same decomposition of  $B$ , namely,  $B = (x_0 \cup z) \cup (x_1 \cup z)$ , and there is but one decomposition,  $A = (x \cup y) \cup (x \cup z)$ , which cannot be mapped to a decomposition of  $B$ . If this (i.e.,  $\pi[y] = z$ ) does not happen, we are done by induction. If this does happen, we know that  $\gamma(H_0) = \gamma(x_0 \cup y)$  has an element in  $y$  (by the argument at the beginning of the proof). As  $f(x_0 \cup y) = f(x_1 \cup z)$ , by Lemmas 3 and 4, both  $H_0 = x_0 \cup y$  and  $H_1 = x_1 \cup z$  have an element in the  $\gamma$ -subset, at the same positions which are mapped onto each other by  $\pi$ . We get that  $\gamma(x_0 \cup z) \cap \gamma(x_1 \cup z)$  has at least  $n - j$  elements, so by our inductive assumption we have at most  $2^j - 1$  decompositions, which gives at most  $2 \cdot (2^j - 1) + 1 = 2^{j+1} - 1$  decompositions of  $A$ .  $\square$

Let  $G \subseteq P$  be a generic subset. Set

$$S = \bigcup \{s : (s, f) \in G\}, \quad F = \bigcup \{f : (s, f) \in G\}.$$

**Lemma 6.** *There is a  $p \in P$  such that  $p \Vdash |S| = \aleph_n$ .*

*Proof.* Otherwise  $\mathbf{1} \Vdash \sup(S) < \omega_n$ . By ccc, there is an ordinal  $\xi < \omega_n$  for which  $\mathbf{1} \Vdash \sup(S) < \xi$ , but this is impossible as there are conditions in  $P_1$  forcing that  $\xi \in S$ .  $\square$

Now we can conclude the proof of the theorem. If  $G$  is generic, and  $p \in G$  with the condition  $p$  of Lemma 6, then in  $V[G]$   $F$  witnesses the theorem by Lemma 5 (for  $j = n$ ) on the ground set  $S$ . As  $|S| = \omega_n$  we can replace it by  $\omega_n$ .  $\square$

### 3. THE GCH RESULT

Set

$$T_n = \frac{1}{2} \sum_{i=1}^n \binom{n+i}{n} \binom{n}{i}.$$

So  $T_1 = 1$ ,  $T_2 = 6$ ,  $T_3 = 31$ . In general,  $T_n$  is asymptotically  $c(3 + 2\sqrt{2})^n / \sqrt{n}$  for some  $c$ .

**Theorem 2 (GCH).** *If  $F : [\omega_n]^{<\omega} \rightarrow \omega$ , then some  $A \in [\omega_n]^{<\omega}$  has at least  $T_n$  decompositions as  $A = H_0 \cup H_1$ ,  $H_0 \neq H_1$ ,  $F(H_0) = F(H_1)$ .*

*Proof.* By the Erdős-Rado theorem (see [2], [3]) there is a set  $\{x_\alpha : \alpha < \omega_1\}$  which is  $(n-1)$ -end-homogeneous, i.e., for some  $g : [\omega_1]^{<\omega} \rightarrow \omega$ , if  $\alpha_1 < \dots < \alpha_k < \beta_1 < \dots < \beta_{n-1} < \omega_1$  then

$$f(\{x_{\alpha_1}, \dots, x_{\alpha_k}, x_{\beta_1}, \dots, x_{\beta_{n-1}}\}) = g(\alpha_1, \dots, \alpha_k).$$

Select  $S_1 \in [\omega_1]^{\omega_1}$  in such a way that  $g(\alpha) = c_0$  for  $\alpha \in S_1$ . Set  $\gamma_1 = \min(S_1)$ . In general, if  $\gamma_i, S_i$  are given ( $1 \leq i < n$ ), pick  $S_{i+1} \in [S_i - (\gamma_i + 1)]^{\omega_1}$  so that  $g(\gamma_1, \dots, \gamma_i, \alpha) = c_i$  for  $\alpha \in S_{i+1}$ , and set  $\gamma_{i+1} = \min(S_{i+1})$ . Given  $\gamma_1, \dots, \gamma_n$  and  $S_n$ , let  $\gamma_{n+1}, \dots, \gamma_{2n}$  be the  $n$  least elements of  $S_n - (\gamma_n + 1)$ .

Our set will be  $A = \{x_{\gamma_1}, \dots, x_{\gamma_{2n}}\}$ . For  $0 \leq i < n$  the color of any  $(n+i)$ -element subset of  $A$  containing  $x_{\gamma_1}, \dots, x_{\gamma_i}$  will be  $c_i$ . We can select  $\frac{1}{2} \binom{2n-i}{n} \binom{n}{i}$  different pairs of those sets which cover  $A$ . In toto, we get  $T_n$  decompositions of  $A$ .  $\square$

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