INDUCING CHARACTERS AND NILPOTENT SUBGROUPS

GABRIEL NAVARRO

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Abstract. If $H$ is a subgroup of a finite group $G$ and $\gamma \in \text{Irr}(H)$ induces irreducibly up to $G$, we prove that, under certain odd hypothesis, $F(G)F(H)$ is a nilpotent subgroup of $G$.

1. Introduction

If some character of a subgroup $H$ of a finite group $G$ induces irreducibly up to $G$, one expects $H$ to be large enough to contain nontrivial information on $G$. In this note, we relate the Fitting subgroups of $H$ and $G$.

Theorem A. Let $H \subseteq G$ and suppose that $\gamma$ is a character of $H$ with $\gamma^G \in \text{Irr}(G)$. If either $|G : H|$ or $|H : F(H)|$ is odd, then $F(G)F(H)$ is a nilpotent subgroup of $G$.

Notice that Theorem A is no longer true without the odd hypothesis, $GL(2,3)$ being a solvable counterexample (we may take $H$ to be a 3-Sylow normalizer).

Theorem A may be applied to study characters induced from nilpotent subgroups, and this is what we do in Section 3 below.

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2. Proof of Theorem A

The proof of our Theorem A relies on the deep facts on odd fully ramified sections discovered by I. M. Isaacs ([1]).

(2.1) Theorem. Let $L \subseteq K \triangleleft G$ with $L \triangleleft G$, $K/L$ abelian, and assume that either $|G : K|$ or $|K : L|$ is odd. Let $\phi \in \text{Irr}(L)$ be invariant in $G$ and assume that $\phi^K = e\theta$ for some $\theta \in \text{Irr}(K)$ and integer $e$. Then there exists $U \subseteq G$ such that

(a) $UK = G$ and $U \cap K = L$,
(b) If $|K/L| > 1$ and $\xi \in \text{Irr}(U|\phi)$, then $\xi^G$ is reducible.

Proof. This is a well known consequence of Theorems (9.1) and (9.2) of [1]. If $\xi^G$ is irreducible, Theorem (9.2) forces the canonical character $\psi$ to be irreducible. However, $\psi \psi$ is the permutation character of $G$ on $K/L$ (see the values of the character $\psi$), and since this action cannot be transitive, $[\psi, \psi] > 1$, a contradiction.
Proof of Theorem A. We argue by double induction, first on $|G|$ and second on $|G : H|$, and we assume that $H$ is proper in $G$.

First, we show that it is no loss to assume that $G = H F(G)$. If we write $F = F(G)$ and $K = H F < G$, since $\gamma^K$ is irreducible, and either $|K : H|$ or $|H : F(H)|$ is odd, by the inductive hypothesis, we have that $F(K)F(H)$ is nilpotent. But then, since $F \subseteq F(K)$, the theorem follows in this case.

We claim that we may also assume that $H$ is a maximal subgroup of $G$. If $H < J < G$, arguing as before, we have that $(F \cap J)F(H)$ is nilpotent. Since $J = (F \cap J)H$, we have that $(F \cap J)F(H) \subseteq F(J)$. Then, we have that either $|G : J|$ or $|J : F(J)|$ is odd. Now, since $(\gamma^J)_G$ is irreducible, by induction on $|G : H|$, we have that $FF(H) \subseteq FF(J)$ is nilpotent, and this proves the claim.

Now, write $L = F \cap H$ and notice that $L$ is normal in $G$, because $L < H$ and $L < N_F(L)$. Since $H$ is maximal (and $F$ is nilpotent), we deduce that $F/L$ is an abelian $p$-chief factor of $G$, for some prime $p$.

Write $F(H) = X \times Y$, where $X$ is the Sylow $p$-subgroup of $F(H)$ and $Y$ its $p$-complement and similarly write $F = F_p \times F'_p$. Since $F/L$ is a $p$-group, then $F_pL = F$ and thus $F_p = H$. Hence, we have that $F(H)/L$ is a $p'$-group, because $F_pX$ is a $p$-group normalized by $H$ and then normal in $G$.

We now start looking at characters. If $\theta \in \text{Irr}(X)$ is a constituent of $\gamma_X$, we claim that $\theta$ is $G$-invariant (recall that $X = F_p \cap L$ is normal in $G$). Certainly, $\theta$ is $Y$-invariant, since $[X, Y] = 1$ and therefore the inertia group $V$ of $\theta$ in $H$ contains $F(H)$. Since $\theta$ lies under $\gamma$, by the Clifford Correspondence there is a character $\delta \in \text{Irr}(V)$ such that $\delta^H = \gamma$. Therefore, $\delta^G$ is irreducible, and so it is $\delta^{FV}$. Now, if $V < H$, then $FV$ is proper in $G$ and we apply the inductive hypothesis to conclude that $F(V)F$ is nilpotent. Since $F(H) \subseteq F(V)$, the theorem is proven in this case. So we assume that $\theta$ is $H$-invariant. Now, since $H$ is maximal in $G$, we have that either $\theta$ is $G$-invariant or its inertia group in $G$ is exactly $H$. So, if $\theta$ is not $G$-invariant, we have that $I_{F_p}(\theta) = H \cap F_p = X$ and thus, we deduce that $\theta^{F_p}$ is irreducible. We now prove that this implies $FF(H)$ to be nilpotent. First, notice that the $p'$-group $Y$ acts on the $p$-group $F_p$ in such a way that $X \subseteq C_{F_p}(Y)$. Since $F_p/X$ is a chief factor of $G$, we have that $C_{F_p}(Y) = X$ or $[F_p, Y] = 1$. Since $FF(H) = F_pY$, we may assume that $C_{F_p}(Y) = X$. Hence, $C_{F_p/X}(Y) = 1$. But then, the $Y$-invariant irreducible character $\theta^{F_p}$ restricted to $X$ has a unique $Y$-invariant irreducible constituent, by Problem (13.4) of [2], for instance. Since $X$ is centralized by $Y$, this implies that $\theta$ is stabilized by $F_p$. Since $\theta$ induces irreducibly up to $F_p$, by Problem (6.1) of [2], we have that $F_p = X$ and thus, that $G = H$, a contradiction. This proves $\theta$ to be $G$-invariant, as claimed.

Now, since $([FY : F_p], [F_p : X]) = 1$ and $C_{F_p/X}(Y) = 1$, we are in the hypotheses of Problem (13.10) of [2]. Hence, we may conclude that there exists a unique $Y$-invariant $\phi \in \text{Irr}(F_p)$ lying over $\theta$. Since $Y < H$, if $h \in H$, notice that $\phi^h$ is $Y$-invariant and lies over $\theta$. Therefore, by uniqueness, we have that $\phi$ is $G$-invariant. Hence, by Mackey, $(\gamma^G)_F$ is a multiple of $\phi$.

We are now ready to apply the Going Down Theorem (6.18) of [2] and we conclude that $\phi_X = \theta$ or that $\phi$ is fully ramified over $\theta$. In the first case by Corollary (4.2) of [3], we have that $(\gamma^G)_H$ is irreducible. This is impossible, unless $H = G$.

So, we may assume that $\phi$ is fully ramified over $\theta$. Now we wish to apply Theorem (2.1) and hence we check that $H$ (up to $G$-conjugacy) is the unique complement of $F_p/X$ in $G$. This follows by realizing that $H/X = N_{G/X}(XY/X)$ together with the
fact that \( XY/X = (FY \cap H)/X \) is a Hall \( p \)-complement of \( FY/X \). Now, since by our hypotheses we have that either \( |F_p : X| \) or \( |H : X| \) is odd, we apply Theorem (2.1) to get the final contradiction.

(2.3) Corollary. Suppose that \( H \subseteq G \) is nilpotent. If \( \gamma \) is a character of \( H \) with \( \gamma^G \) irreducible, then \( \mathbf{F}(G)H \) is nilpotent.

Proof. In this case, \( |H : \mathbf{F}(H)| \) is odd and Theorem A applies.

3. Characters induced from nilpotent subgroups

In this section, we associate to any \( \chi \in \text{Irr}(G) \) a uniquely defined (up to conjugacy in \( G \)) pair \( (S, \sigma) \), where \( S \) is a subgroup of \( G \) and \( \sigma \in \text{Irr}(S) \) induces \( \chi \). For solvable groups, we will prove that \( \chi \) is induced from a nilpotent subgroup if and only if \( S \) is nilpotent. In other words, whenever \( \chi \in \text{Irr}(G) \) can be obtained via induction from a nilpotent subgroup, this can be done in a standard way.

Before introducing the pair \( (S, \sigma) \) we need to derive a (perhaps) surprising consequence of (2.3).

(3.1) Theorem. Let \( G \) be a solvable group and suppose that \( \chi \in \text{Irr}(G) \) is induced from a nilpotent subgroup. If \( \chi_{\mathbf{F}(G)} \) is homogeneous, then \( G \) is nilpotent.

Proof. Write \( \chi = \gamma^G \), where \( \gamma \in \text{Irr}(H) \) and \( H \) is maximal with respect to being nilpotent. By Corollary (2.3), we have that \( F = \mathbf{F}(G) \subseteq H \). Now, let \( M/F = \mathbf{F}(G/F) \) and notice that \( M \cap H = F \). This is because \( F \subseteq M \cap H \subseteq M \) and hence, \( M \cap H \) is both nilpotent and subnormal in \( G \).

Write \( \chi_F = e\theta \), where \( \theta \in \text{Irr}(F) \) and notice that, in the notation of [2], we have that \( (G,F,\theta) \) is a character triple. By Theorem (11.28) of [2], we may find \( (G^*,F^*,\theta^*) \) is an isomorphic character triple with \( F^* \subseteq Z(G^*) \). Therefore, observe that \( M^* = \mathbf{F}(G^*) \) (we use the notation \( (M/F)^* = M^*/F^* \), where \( * \) also denotes the group isomorphism between \( G/F \) and \( G^*/F^* \)). Since \( H^* \) is also nilpotent and \( (\gamma^*)^G \) is irreducible (because \( \gamma^G \) is), we may again apply Corollary (2.3) to conclude that \( M^*H^* \) is nilpotent. Hence, the group \( MH/F \) is also nilpotent. But in this case, since \( F \subseteq H \), we have that \( H \) is nilpotent and subnormal in \( MH \). Thus \( H \subseteq \mathbf{F}(MH) \) and by the maximality of \( H \), we conclude that \( H = \mathbf{F}(MH) \triangleleft MH \). Now, since \( M \cap H = F \), we have that \( H/F \subseteq C_{G/F}(M/F) \). Since \( G \) is solvable, the group \( M/F \) contains its own centralizer and thus, we have that \( H/F \subseteq M/F \).

This implies \( H = F \) and \( \theta = \gamma \). Now, since \( \theta \) is \( G \)-invariant and induces irreducibly up to \( G \), we conclude that \( F = G \), as required.

If \( G \) is a finite group and \( \chi \in \text{Irr}(G) \), we are going to define a uniquely determined (up to \( G \)-conjugacy) pair \( (S, \sigma) \) associated to \( \chi \), as follows. Choose \( \theta \in \text{Irr}(\mathbf{F}(G)) \) to be any irreducible constituent of \( \chi_{\mathbf{F}(G)} \), and let \( \mu \in \text{Irr}(T/\theta) \) be the Clifford correspondent of \( \chi \) over \( \theta \). If \( T = G \), we define \( (S, \sigma) = (G, \chi) \). On the other hand, if \( T < G \), we inductively define \( (S, \sigma) \) for \( \chi \) to be the corresponding \( (S, \sigma) \) for \( \mu \).

Notice that \( (S, \sigma) \) is determined up to conjugacy in \( G \) and that it satisfies: \( \mathbf{F}(G) \subseteq S \), \( \sigma^G = \chi \) and \( \sigma_{\mathbf{F}(S)} \) is homogeneous.

(3.2) Theorem. Let \( G \) be a solvable group and let \( \chi \in \text{Irr}(G) \). Then \( \chi \) is induced from a nilpotent subgroup if and only if \( S \) is nilpotent.

Proof. Certainly, if \( S \) is nilpotent, then \( \chi \) is induced from a nilpotent subgroup so we prove the converse. We argue by induction on \( |G| \). Let \( F = \mathbf{F}(G) \), let \( \theta \in \text{Irr}(F) \)
be under $\chi$ and let $\mu \in \text{Irr}(T)$ be the Clifford correspondent of $\chi$ over $\theta$, so that the pair $(S, \sigma)$ for $\mu$ is also a pair for $\chi$.

Write $\chi = \gamma^G$, where $\gamma \in \text{Irr}(H)$ and $H$ is nilpotent. By Corollary (2.3), we may assume that $F \subseteq H$ and by replacing $\theta$ by some $G$-conjugate, we may assume that $\gamma$ lies over $\theta$. Now, let $\tau \in \text{Irr}(T \cap H|\theta)$ be the Clifford correspondent of $\gamma$ over $\theta$ and notice that $\tau^G = \chi$. By uniqueness of Clifford correspondents, we have that $\tau^T = \mu$. So we have that $\mu$ is also induced from a nilpotent subgroup. Therefore, if $T$ is proper in $G$, by induction, $S$ is nilpotent and we are done in this case. If $T = G$, by Theorem (3.1), $G$ is nilpotent and the result follows. \qed

References


Departament d’Algebra, Facultat de Matematiques, Universitat de Valencia, 46100 Burjassot, Valencia, Spain

E-mail address: gabriel@vm.ci.uv.es