ON THE COMMUTANT OF HYPONORMAL OPERATORS

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Abstract. Let $T$ be a pure hyponormal operator with compact self-commutator. We show that the unit ball of the commutant of $T^*$ is compact in the strong operator topology.

Let $H$ be a separable complex Hilbert space and let $L(H)$ denote the algebra of all bounded linear operators on $H$. An operator $T \in L(H)$ is hyponormal if $T^*T - TT^* \geq 0$ and essentially normal if $T^*T - TT^*$ is compact. Moreover $T$ is pure if it has no nonzero reducing subspace on which it is normal. The main result of this paper is the following:

**Theorem 1.** Let $T \in L(H)$ be a pure hyponormal essentially normal operator. Then the closed unit ball of the commutant algebra $\{T^*\}'$ is compact in the strong operator topology.

The proof of this theorem relies on the following two results. The first one is a particular case of a more general result obtained by V. Lomonosov.

**Theorem 2** (cf. [L, Lemma 1]). Let $T \in L(H)$ be an essentially normal operator. Suppose there exists a sequence $\{T_n\}$ in the commutant of $T$ such that $T_n \to 0$ weakly but $T_n \not\to 0$ strongly. Then there exist a normal operator $N \in L(H)$ and a nonzero operator $X \in L(H)$ such that $NX = XT$.

This theorem has been used in [L] to show that for a transitive essentially normal operator $T \in L(H)$ either $\{T\}'$ or $\{T^*\}'$ has strongly compact ball. The second result we need is an asymmetric Fuglede-Putnam theorem due to J.G. Stampfli and B.L. Wadhwa.

**Theorem 3** (cf. [SW, Theorem 1]). Let $T \in L(H)$ be a hyponormal operator and suppose there exist a normal operator $N \in L(H)$ and $X \in L(H)$ with dense range, such that $TX = XN$. Then $T$ is normal.

**Proof of Theorem 1.** Suppose the conclusion is false. Now, recall that the strong topology is metrizable on bounded subsets of $L(H)$ and that the unit ball of $L(H)$ is metrizable and compact in the weak operator topology. Therefore there exists a sequence of operators $\{T_n\}$ in $\{T^*\}'$ such that $T_n \to 0$ weakly but $T_n \not\to 0$ strongly. An application of Theorem 2 above yields a normal operator $N \in L(H)$ and a nonzero operator $X \in L(H)$ such that $NX = XT^*$. Therefore $TX^* = X^*N^*$.
and the restriction $S$ of $T$ to $\overline{TmX^*}$ is still hyponormal. Applying Theorem 3 we infer that $S$ is normal, which contradicts our initial assumption that $T$ is pure hyponormal. The proof is complete.

The following is an easy consequence of Theorem 1.

**Corollary 1.** Under the assumptions of Theorem 1, the weak*, weak and strong operator topologies agree on bounded subsets of $\{T^*\}'$.

For a compact set $K \subset C$ let $Rat(K)$ denote the algebra of all rational functions with poles off $K$. An operator $T \in L(H)$ is multicyclic if there exists a finite family of vectors $\{x_1, \ldots, x_m\}$ in $H$ such that the space $\{\sum_{k=1}^m f_k(T)x_k; f_1, \ldots, f_m \in Rat(\sigma(T))\}$ is dense in $H$.

**Corollary 2.** Let $T \in L(H)$ be a multicyclic pure hyponormal operator. Then the weak*, weak and strong operator topologies agree on bounded subsets of $\{T^*\}'$.

**Proof.** Since $T$ is multicyclic hyponormal, it is also essentially normal by a celebrated theorem of Berger and Shaw [BS]. Now, an application of Theorem 1 yields the conclusion.

**References**


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