COMBINATORICS OF A CERTAIN IDEAL IN THE SEGRE COORDINATE RING

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Abstract. We focus on a “fat” model of an ideal in the class of the canonical ideal of the Segre coordinate ring, looking at its Rees algebra and related arithmetical questions.

1. Introduction

Let $\mathcal{S}$ be the image of the Segre map
$$\sigma = \sigma_{n-1,m-1} : \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \longrightarrow \mathbb{P}^{nm-1},$$
the so-called Segre variety. As a toric variety, $\mathcal{S}$ admits $k[t_{i s_j}](1 \leq i \leq n, 1 \leq j \leq m)$ as coordinate ring. This ring can be presented over the polynomial ring $k[X] = k[X_{ij}](1 \leq i \leq n, 1 \leq j \leq m)$ by the ideal $I_2(X_{ij})$ generated by the $2 \times 2$ minors of the generic $n \times m$ matrix $(X_{ij}).$ It is well known that the canonical class of the latter is $(m-n)[K],$ where $K \subset S = k[X]/I_2(X_{ij})$ is the ideal generated by (the residues of) the entries in the first column of the matrix $(X_{ij})$ (cf. [BV], (8.4)).

Now, given an integer $d \geq 1,$ let $R^{(d)}$ denote the ideal generated by the $d$th powers of the generators of $K.$ The main purpose of this paper is to investigate the algebraic-combinatorics of the blowup of $\mathcal{S}$ along the locus of $R^{(d)}.$ Algebraically, we are therefore looking at the Rees algebra of the ideal $R^{(d)}.$ Using the toric representation, this algebra is simply the $k$-subalgebra
$$k[t_{i s_j}, (t_1 s_1)^d T, \ldots, (t_n s_1)^d T] \subset k[t, s][T],$$
where $1 \leq i \leq n, 1 \leq j \leq m.$ Since $s_1$ is fixed in the $d$th powers, it is not difficult to see that this algebra is isomorphic to the $k$-algebra $R^{(d)} = k[t_{i s_j}, t_1^d, \ldots, t_n^d] \subset k[t, s].$

As it turns out, $R^{(d)}$ is presented over a polynomial ring $A = k[X, U],$ with $X = \{X_{ij}\}, U = \{U_1, \ldots, U_n\},$ by a sum of determinantal ideals, each generated by certain $2 \times 2$ minors, so our toric variety is a sort of determinantal locus lacking the generic codimension. It can be looked at as the generic version of a few classes of ideals appearing in the recent literature (cf. [Hu], [HuHu], [Sch] and [MoSi]), obtained thereof by specialization and by taking suitable free ring extensions.
2. A pseudo-determinantal locus

We will fix the following notation:

- $I_r(L)$: the ideal generated by $r \times r$ minors of the matrix $L$.
- $t, s$: sets of (toric) variables $t_1, \ldots, t_n, s_1, \ldots, s_m$ over a field $k$.
- $S$: the coordinate ring $k[x_{ij}] = k[X_{ij}]/I_2(X_{ij})$ of the Segre embedding.
- $\mathcal{R}^{[d]}$: the ideal (row-matrix) in $S$ generated by the $d$th powers of $x_{11}, \ldots, x_{nm}$.
- $R^{[d]}$: the toric ring $k[t_{ij}, t_1^d, \ldots, t_n^d]$.
- $M(Y)$: a monomial in the variables $Y$.
- $M(Y)$: the residue of the monomial $M(Y)$ modulo some ideal.
- $\mathbb{M}(d, Y)$: the set (row, ideal) of all monomials of degree $d$ in the variables $Y$.
- $\mathbb{M}(d, y)$: the set of residues of $\mathbb{M}(d, Y)$.

2.1. The defining equations. One needs the following lemmata. In order to save on notation, we set sometimes $X_i = X_{1i}, \ldots, X_{mi}$ and, correspondingly, $x_i = x_{1i}, \ldots, x_{mi}$.

(2.1.1) Lemma. For any pair of indices $1 \leq i_1, i_2 \leq n$, consider the involutive $k$-algebra automorphism $\Phi = \Phi_{i_1, i_2}$ of the polynomial ring $k[X_{ij}] = k[X_1, \ldots, X_n]$ such that

$$
\Phi(X_{i,j}) = \begin{cases} 
X_{i_2,j} & \text{if } i = i_1, \\
X_{i_1,j} & \text{if } i = i_2, \\
x_{i,j} & \text{otherwise}.
\end{cases}
$$

Then:

(i) $\Phi$ induces an automorphism of $S = k[X_{ij}]/I_2(X_{ij})$.
(ii) For any two monomials $M = M(X_{i_1}), N = N(X_{i_1}) \in k[X_{i_1}]$ of the same degree, one has $M\Phi_{i_1, i_2}(N) \equiv N\Phi_{i_1, i_2}(M) \pmod{I_2(X_{ij})}$.

Proof. (i) Clearly, the ideal $I_2(X_{ij})$ is invariant under $\Phi$. Since $\Phi$ is an involution (i.e., $\Phi = \Phi^{-1}$), it then induces an automorphism of $S$.

(ii) One proceeds by induction on the common degree of $M$ and $N$. The result is trivial if $M = N$, so assume these are distinct monomials. Now write $M = X_{i_1,j_1}M_1$ and $N = X_{i_2,j_2}N_1$, with $j_1 \neq j_2$. Then, with $\Phi = \Phi_{i_1, i_2}$ and by the inductive hypothesis:

$$
M\Phi(N) = X_{i_1,j_1}X_{i_2,j_2}M_1\Phi(N_1) \equiv X_{i_1,j_2}X_{i_2,j_1}M_1\Phi(N_1)
\equiv X_{i_1,j_2}X_{i_2,j_1}N_1\Phi(M_1) = N\Phi(M),
$$

as required. \qed

(2.1.2) Remark. Part (ii) of Lemma (2.1.1) has been used before in different forms (cf., e.g., [Gim, Lemme 5.12.1]).

(2.1.3) Lemma. The first syzygies of the ideal $\mathcal{R}^{[d]} \subset S$ are generated by the first syzygies of all pairs $\{x_{i_1,1}^d, x_{i_2,1}^d\}$, $1 \leq i_1, i_2 \leq n$ and these are generated by those syzygies whose coordinates are terms $\alpha M$, $\alpha \in k$ and $M$ a monomial.

Proof. This is a direct consequence of the fact that $S$ is defined by a binomial ideal [EiSt, Corollary 1.7 (b)]. \qed
Here is the basic technical result of this section:

(2.1.4) Proposition. Let $d \geq 1$. The ideal $\mathfrak{R}^{[d]} \subset S$ has the following presentation as an $S$-module:

$$
\bigwedge^2 \left( S^n \right)_{\mathfrak{C}(m,d)} \xrightarrow{\psi^{[d]}} S^n \xrightarrow{\mathfrak{R}^{[d]}} S,
$$

where $\mathfrak{C}(m,d) = \begin{pmatrix} m-1+d \end{pmatrix}_d$, $\mathfrak{R}^{[d]}$ stands for the map given by the row-matrix $(x_1^d \ldots x_n^d)$ and $\psi^{[d]}$ is given by the matrix

$$
\begin{pmatrix}
-\mathfrak{M}(d,x_2) & -\mathfrak{M}(d,x_3) & \cdots & -\mathfrak{M}(d,x_n) \\
\mathfrak{M}(d,x_1) & 0 & \cdots & 0 \\
0 & \mathfrak{M}(d,x_1) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mathfrak{M}(d,x_1) \\
\end{pmatrix} \rightarrow 
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
-\mathfrak{M}(d,x_1) & 0 & \cdots & 0 \\
\mathfrak{M}(d,x_2) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mathfrak{M}(d,x_2) \\
\end{pmatrix}.
$$

Proof. The containment $\text{Im} \psi^{[d]} \subset \ker \mathfrak{R}^{[d]}$ is a straightforward consequence of Lemma (2.1.1), (ii).

For the reverse inclusion, by Lemma (2.1.3) and by an obvious symmetrical argument we may assume that we are given a relation of the form

$$
(2.1)
X_1^d M + X_2^d N \equiv 0 \pmod{I_2(X_{ij})},
$$

where $M$ and $N$ are terms in $k[\mathbf{X}]$.

The crucial point is to establish that $\deg_{\mathbf{X}_2} M \geq d$. At any rate, one has $\deg_{\mathbf{X}_2} M \geq 1$, otherwise by setting to zero all the variables in $\mathbf{X}_2$, it would follow that $X_1^d M \in I_2(X_{ij})(i \neq 2)$ which is absurd.

We proceed by induction on $d$, the assertion for $d = 1$ having just been shown. Thus, let $d \geq 2$ and assume that $\deg_{\mathbf{X}_2} M = d_0 < d$. By the preceding, $d_0 \geq 1$, hence $d - d_0 \leq d - 1$. Write $M = \overline{M}_1 M_1$, with $\overline{M} \in \mathfrak{M}(d_0, \mathbf{X}_2)$. By Lemma (2.1.1), (ii), one has $X_1^d \overline{M} = X_1^d \Phi_1(\overline{M})$, hence (2.1) yields

$$
-X_1^d N \equiv X_1^d X_1^d \overline{M}_1 M_1 \equiv X_1^d X_1^d \Phi_1(\overline{M}) M_1,
$$

from which it follows that $X_1^d \Phi_1(\overline{M}) M_1 \equiv X_1^d \Phi_1(\overline{M}) M_1$.

Thus, we can write $M = \overline{M} M_1$, where $\overline{M} \in \mathfrak{M}(d, \mathbf{X}_2)$. By Lemma (2.1.1), (ii), we have $X_1^d \overline{M} = X_1^d \Phi_1(\overline{M})$, from which it follows that $X_1^d M \equiv X_1^d \Phi_1(\overline{M}) M_1$.

Using (2.1), one then obtains $X_1^d \Phi_1(\overline{M}) M_1 + N \equiv 0$, hence $N \equiv -\Phi_1(\overline{M}) M_1$ because $I_2(X_{ij})$ is a prime ideal. Since $\overline{M} \in \mathfrak{M}(d, \mathbf{X}_2)$, it follows that $\Phi_1(\overline{M}) \in \mathfrak{M}(d, \mathbf{X}_2)$.

Altogether, one gets

$$
\begin{pmatrix}
M(\mathbf{x}) \\
N(\mathbf{x})
\end{pmatrix} = M_1(\mathbf{x}) \begin{pmatrix}
\overline{M}(\mathbf{x}_2) \\
-M(\mathbf{x}_1)
\end{pmatrix} \in \text{Im} \psi^{[d]},
$$

as was to be shown.

Here is the main result of this section:

(2.1.5) Theorem. Let $d \geq 1$ be an integer and let $\mathfrak{R}^{[d]} \subset S = k[\mathbf{X}] / I_2(X_{ij})$ as before stand for the ideal generated by the $d$th powers of the generators of the ideal
\( \mathfrak{R} \) of \( S \). Also let \( R^{[d]} = k[t_{ij}^{d_1}, \ldots, t_{ij}^{d_m}] \subset k[t, s] \) (\( 1 \leq i \leq n, 1 \leq j \leq m \)). Then:

(i) \ The ideal \( R^{[d]} \) is of linear type.

(ii) \ There is a presentation

\[
R^{[d]} \simeq k[X_1, X_2, \ldots, X_n, U]/ \sum_{1 \leq i_1 < i_2 \leq n} I_2(L_{i_1, i_2})
\]

where

\[
L_{i_1, i_2} = \left( \begin{array}{c}
X_{i_1, 1} \ldots X_{i_1, m} \\
X_{i_2, 1} \ldots X_{i_2, m}
\end{array} \right) \frac{U_{i_1} \cdot M(d - 1, X_{i_1})}{U_{i_2} \cdot M(d - 1, X_{i_2})},
\]

with \( U_{i_1} \cdot M(d - 1, X_{i_1}) \) designating the row whose entries are the entries of \( M(d - 1, X_{i_1}) \) multiplied by the variable \( U_{i_1} \).

Proof. (i) We show that the generators \( x_{11}^{d}, \ldots, x_{n1}^{d} \) of \( \mathfrak{R}^{[d]} \) form a \( d \)-sequence. For that, we use the characterization of such sequences as given in [HSV, Section 6] to the effect that

\[
((x_{11}^{d}, \ldots, x_{s1}^{d})): x_{s+1}^{d}) \cap \mathfrak{R}^{[d]} = (x_{11}^{d}, \ldots, x_{s1}^{d}) \quad \text{for} \quad 0 \leq s \leq n - 1.
\]

By Proposition (2.1.4), one sees that

\[
((x_{11}^{d}, \ldots, x_{s1}^{d}): x_{s+1}^{d}) = (M(d, x_1), M(d, x_2), \ldots, M(d, x_s)),
\]

hence we are to prove that

\[
(M(d, x_1), M(d, x_2), \ldots, M(d, x_s)) \cap (x_{11}^{d}, \ldots, x_{s1}^{d}) \subset (x_{11}^{d}, \ldots, x_{n1}^{d}).
\]

Set \( J_1 = (M(d, x_1), M(d, x_2), \ldots, M(d, x_s)) \) and \( J_2 = (x_{11}^{d}, \ldots, x_{n1}^{d}) \).

To compute the above intersection of monomial ideals modulo the binomial ideal \( I_2(X_{ij}) \) we follow the prescription given in [EiSt, Proof of Corollary 1.6]: choose a monomial order on the polynomial ring \( A = k[X_{ij}] \) and take the standard monomials mod \( I_2(X_{ij}) \); then, \( M(x) \subset A/I_2(X_{ij}) \), the set of residues of the standard monomials, is a vector space basis of \( A/I_2(X_{ij}) \); next, one takes a vector space basis \( J_1 \) (resp. \( J_2 \)) of \( J_1 \) (resp. \( J_2 \)) mod \( I_2(X_{ij}) \) which is contained in \( M(x) \); at the outset, \( J_1 \cap J_2 \) is a vector space basis of the ideal \( J_1 \cap J_2 \).

Now, in the present case, choosing a suitable order, the \( 2 \times 2 \) minors already form a Gröbner basis of the ideal \( I_2(X_{ij}) \) (cf., e.g., [Stu]). Therefore, a monomial in \( M(x) \) is characterized by the property that it involves indeterminates belonging to one and only one row or to one and only one column of the matrix \( (x_{ij}) \). It follows from this that

\[
J_1 = \bigcup_{1 \leq i \leq s \atop r \geq d} M(r, x_i) \cup \{x_{ij}^{d_i}M(x_i') \mid 1 \leq i \leq s, 1 \leq j \leq m \}
\]

is a vector basis of \( J_1 \), where \( M(x_i') \) designates a monomial involving only variables along the \( j \)th column.

By a similar token,

\[
J_2 = \{x_{i1}^{d_i}M(x_i'), x_{i1}^{d_i}M(x_i) \mid 1 \leq i \leq n \}
\]

is a vector basis of \( J_2 \), where \( M(x_i) \) designates a monomial involving only variables along the \( i \)th row. One clearly has \( J_1 \cap J_2 = \{x_{i1}^{d_i}M(x_i'), x_{i1}^{d_i}M(x_i) \mid 1 \leq i \leq s \} \). Therefore, the ideal \( J_1 \cap J_2 \) is generated by \( \{x_{i1}^{d_i} \mid 1 \leq i \leq s \} \), as was to be shown.
(ii) By part (i), the canonical surjection \( S(\mathfrak{R}^{[d]}) \rightarrow \mathcal{R}(\mathfrak{R}^{[d]}) \) is an isomorphism, where \( S(\mathfrak{R}^{[d]}) \) and \( \mathcal{R}(\mathfrak{R}^{[d]}) \) denote the symmetric and the Rees algebra of the ideal \( \mathfrak{R}^{[d]} \), respectively. On the other hand, by Proposition (2.1.4), \( S(\mathfrak{R}^{[d]}) \) admits the presentation that is being proposed for \( R^{[d]} \). Therefore, it suffices to show that \( R^{[d]} \) is isomorphic to \( \mathcal{R}(\mathfrak{R}^{[d]}) \). Clearly,
\[
\mathcal{R}(\mathfrak{R}^{[d]}) \cong S[\mathfrak{R}^{[d]} T] \cong k[t_1 s_1, (t_1 s_1)^d T, \ldots, (t_n s_1)^d T] \subset k[t, s][T].
\]
Since \( s_1^d \) is a common factor throughout the terms \( t^d s_1^d T \) and these have a fixed degree, we see that there is an isomorphism \( k[t, s, t_1^d, \ldots, t_n^d] \cong k[t s_1, (t_1 s_1)^d T, \ldots, (t_n s_1)^d T] \).

\[\square\]

2.2. Hilbert function data of \( R^{[d]} \). The reader is referred to [HUT] and [STV] for the background needed in this portion. Again, one considers the Segre ring \( S = k[\mathbf{X}]/I_2(\mathbf{X}) \), which will be thought of as the current base ring. By Theorem (2.1.5), \( R^{[d]} \) is isomorphic to the Rees algebra of the ideal \( \mathfrak{R}^{[d]} \subset S \) and, moreover, as such, it has a natural structure of standard bigraded \( k \)-algebra, its presentation ideal over \( S \) being bihomogeneous with respect to the two sets of variables \( \mathbf{X} = \{X_{ij}\} \) and \( \mathbf{U} = \{U_1, \ldots, U_n\} \).

Consider an \( \mathbb{N}^{n+1} \)-gradation on \( S[\mathbf{U}] \) by setting
\[
S[\mathbf{U}]_{[a_0, a_1, \ldots, a_n]} := S_{a_0} U_1^{a_1} \cdots U_n^{a_n}.
\]
Let \( \succeq \) be the graded lexicographic order on the monoid \( \mathbb{N}^{n+1} \). It induces a filtration \( \mathcal{F} \) on \( S[\mathbf{U}] \), with \( \mathcal{F}_a := \oplus_{b \succeq a} S[\mathbf{U}]_b \), hence also on the residue ring \( R^{[d]} \cong S[\mathbf{U}]/\mathcal{J} \) which we still denote by \( \mathcal{F} \). Letting \( \mathcal{J}^* \) denote the ideal generated by the initial forms of \( \mathcal{J} \), one has \( \text{gr}_{\mathcal{F}}((R^{[d]})) \cong S[\mathbf{U}]/\mathcal{J}^* \) as bigraded \( k \)-algebras.

By Proposition (2.1.4) (or by the proof of Theorem (2.1.5), (i)) and [HUT, Lemma 1.1], one obtains
\[
\mathcal{J}^* = (M(d, x_1) U_2, (M(d, x_1), M(d, x_2)) U_3, \ldots, (M(d, x_1), \ldots, M(d, x_{n-1})) U_n).
\]

(2.2.1) **Proposition.** With the preceding notation and considering \( R^{[d]} \) and \( \text{gr}_{\mathcal{F}}((R^{[d]})) \) as \( \mathbb{N} \)-graded rings (via the homomorphism \( \mathbb{N}^2 \rightarrow \mathbb{N}, (a, b) \mapsto a + b \)), one has:

(i) \( R^{[d]} \) and \( \text{gr}_{\mathcal{F}}((R^{[d]})) \) admit the same Hilbert function.

(ii) The multiplicity of \( R^{[d]} \) is
\[
e(R^{[d]}) = \sum_{j=0}^{n-1} d^j \frac{m+n-j-2}{n-j-1}.
\]

**Proof.** (i) This is easy and holds quite generally.

(ii) We apply [HUT, Theorem 1.4] (or rather, its recipe), for which we first check its hypotheses. In the present situation, they boil down to the equalities
\[
dim S/I_j = \dim S - j, \quad 1 \leq j \leq n - 1,
\]
where \( I_j = (M(d, x_1), \ldots, M(d, x_j)) \). To verify these, we show that \( \text{ht} I_j = j \) for \( 1 \leq j \leq n - 1 \) (recalling that \( S \) is Cohen–Macaulay). For every such \( j \), consider the prime ideal
\[
P_j = \{X_{kl} \mid 1 \leq k \leq j, 1 \leq l \leq m \} + I_2(\{X_{kl} \mid j + 1 \leq k' \leq n, 1 \leq l \leq m \})
\]
\[= \{X_1, \ldots, X_j \} + I_2(X \setminus \{X_1, \ldots, X_j \}) \subset k[\mathbf{X}].\]
Clearly, \( P_j S \) is a prime as well and contains \( I_j \). It follows that

\[
\operatorname{ht} I_j \leq \operatorname{ht} P_j S = \operatorname{ht} P_j - \operatorname{ht} I_2(X)
\]

\[
= \operatorname{ht} \{X_1, \ldots, X_j\} + \operatorname{ht} I_2(X \setminus \{X_1, \ldots, X_j\}) - (n-1)(m-1)
\]

\[
= jm + (n-j-1)(m-1) - (n-1)(m-1) = j
\]

On the other hand, it is easy to see that every prime ideal of \( S \) containing \( I_j \) already contains \( P_j S \). This leads to \( \operatorname{ht} I_j = j \), as required.

We now compute the multiplicity \( e(S/I_j) \) by the associativity formula. By the above calculation, this formula reduces to

\[
e(S/I_j) = \ell(S_{P_j S}/I_{P_j S})e(S/P_j S).
\]

To simplify the notation, set \( P = P_j, I = I_j \). Observe that the ideal \( P S/P S \) is generated by the images of the variables \( X_{11}, \ldots, X_{j1} \). Indeed, typically, \( X_{kl} - X_{kl}X_{n1} \equiv 0 \pmod{I_2(X)} \). Since \( X_{n1} \) is invertible, the image of \( X_{kl} \) belongs to the ideal generated by the image of \( X_{kl} \), for \( 1 \leq k \leq j \). The above length is then given by the number of monomials \( \{X_{11}^{a_1} \cdots X_{j1}^{a_j} | 0 \leq a_k \leq d-1, 1 \leq k \leq j \} \). This number is clearly \( d^j \).

Next, one has \( S/P S = k[X]/P \simeq k[X \setminus \{X_1, \ldots, X_j\}]/I_2(X \setminus \{X_1, \ldots, X_j\}) \), which is a Segre ring of size \( (n-j) \times m \). Therefore, \( e(S/P) = (m+n-j-2) \) by a well-known formula (cf., e.g., [STV, Remark 2.5]).

To piece everything together, [HUT, Theorem 1.4] tells us that \( e(R^{[d]}) = \sum_{j=0}^{n-1} e(S/I_j) \), hence we are through.

(2.2.2) Remark. By Proposition (2.2.1), (i), it is in principle possible to compute the Hilbert function of \( R^{[d]} \), but it is hardly the case that it may be of any usefulness here. Thus, for example, \( \dim R^{[d]} = m + n \) follows directly from the fact that \( R^{[d]} \) is a Rees algebra of an ideal in the \( m + n - 1 \)-dimensional domain \( S \).

3. THE DEFINING EQUATIONS OF THE SPECIAL ALGEBRA

As above, let \( I = I^{[d]} \subset k[X, U] \) denote the presentation ideal of the \( k \)-algebra \( R^{[d]} \) and let \( \tilde{I} = IS[U] \subset S[U] \), an ideal generated in bidegree \((d, 1)\). We consider the Rees algebra \( R_{S[U]}(\tilde{I}) \): geometrically, one is looking at the blowup of the product \( S \times P_{U}^{n-1} \) along the subvariety \( Bf_{K}(S) \), where \( K \) denotes the subvariety of \( S \) defined by the ideal \( \tilde{I}^{[d]} \).

The special algebra (or fiber cone algebra) of an ideal (resp. homogeneous ideal) \( a \) in a local (resp. positively graded) ring \( A \) is the residue ring \( F(a) := R_{a}/mR_{a}(a) \), with \( m \) standing for the maximal (resp. maximal graded) ideal of \( A \).

We will take \( A = S[U] \) and \( a = \tilde{I} \). As it will turn out, \( F(\tilde{I}) \) is a nice determinant locus which, in the case where \( n = 2 \), is the coordinate ring of a Veronese variety. The reason for that is a far more reaching principle which may have an independent interest outside the scope of the present work.

(3.1) Theorem. Let \( X, Y \) be mutually independent sets of variables over a field \( k \) of characteristic zero, with \( X \) and \( Y \) having the same number of elements, and let \( f_1, \ldots, f_r \) be homogeneous polynomials in the \( X \)-variables, of the same degree. Let \( U, V \) be two additional variables and set \( A = k[f_1 V - \Phi(f_1) U, \ldots, f_r V - \Phi(f_r) U] \subset k[X, U, V] \).
Let \( k[X, Y, U, V] \), where \( \Phi \) as in Lemma (2.2.1) denotes the involutive \( k \)-isomorphism \( X_i \mapsto Y_i \). Then

\[
 k[f_1, \ldots, f_r] \simeq A/A \cap I_2(X, Y) k[X, Y, U, V]
\]
as graded \( k \)-algebras, where \( I_2(X, Y) \) denotes the ideal of \( k[X, Y] \) generated by the \( 2 \times 2 \) minors of the generic matrix whose rows are \( X \) and \( Y \).

**Proof.** Let \( T_1, \ldots, T_r \) be presentation variables over \( k \) for both algebras. It will suffice to show that they have the same presentation ideal. We show, namely, that any homogeneous polynomial relation of one of the two algebras is a polynomial relation of the other. We need the notion of polarization.

Consider a polynomial ring \( k[T, U] \) in two sets of indeterminates \( T = T_1, \ldots, T_r \) and \( U = U_1, \ldots, U_r \). Clearly, \( k[T, U] \) is a free \( k[U] \)-module with basis the monomials in \( T \).

**Definition.** The polarization of \( T \) by \( U \) is the (unique) \( k[U] \)-homomorphism \( P \) of the \( k[U] \)-module \( k[T, U] \) such that \( P(1) = 0 \) and

\[
P(T^a) = \sum_{a_j \neq 0} a_j U_j T_1^{a_1} \cdots T_r^{a_r}
\]
for \( T^a = T_1^{a_1} \cdots T_r^{a_r} \).

One sets \( P_0(T^a) = T^a \) and \( P_{l+1}(T^a) = P_l(P(T^a)) \). Next, consider the \( k \)-algebra homomorphism \( \Psi' : k[T, U] \to k[f_1, \ldots, f_r, \Phi(f_1), \ldots, \Phi(f_r)] \) such that \( \Psi'(T_j) = f_j, \Psi'(U_j) = \Phi(f_j) \), and let \( \Psi \) denote the restriction of \( \Psi' \) to \( k[T] \).

Let \( F(T) = \sum a_a T^a \in k[T] \) be a homogeneous polynomial of degree \( t \), with \( a = (a_1, \ldots, a_r) \), \( |a| = t \) and \( T^a = T_1^{a_1} \cdots T_r^{a_r} \), and let \( s \) denote the common degree of the \( f \)'s. We claim that \( X_1^s \Psi(P(F(T))) \equiv t Y_1^s \Psi(F(T)) \pmod{I_2(X, Y)} \). Indeed, it follows from Lemma (2.2.1) that, for a given term \( a_a T^a \) of \( F(T) \) \((a_a \neq 0)\), one has

\[
\Psi(P(T^a)) = (a_i(a) + \ldots + a_r) \Phi(f_{i(a)}) f_{i(a)}^{a_{i(a)}-1} f_{i(a)+1}^{a_{i(a)+1}} \cdots f_r^{a_r},
\]
where \( a_i(a) \neq 0, a_i = 0 \) \((i < i(a))\) \((\text{mod } I_2(X, Y))\). By summing up over all terms of \( F(T) \), one obtains

\[
\Psi(P(F(T))) \equiv t \sum a_a \Phi(f_{i(a)}) f_{i(a)}^{a_{i(a)}-1} f_{i(a)+1}^{a_{i(a)+1}} \cdots f_r^{a_r} \pmod{I_2(X, Y)}.
\]
Again by Lemma (2.2.1), one has \( X_1^s \Phi(f_j) = Y_1^s f_j \). Substituting yields the desired result.

Next, by iterating the polarization, one easily gets

\[
(3-1) \quad X_1^l \Psi'(P_l(F(T))) = \frac{t!}{(t-l)!} Y_1^l \Psi(F(T)) \pmod{I_2(X, Y)}
\]
where \( P_l(F(T)) = 0 \) if \( l > t \).

On the other hand, a computation yields

\[
F(f_1 V - \Phi(f_1) U, \ldots, f_r V - \Phi(f_r) U) = \sum_{l=0}^{t} (-1)^l \Psi'(P_l(F(T))) V^{t-l} U^l.
\]
Using (3-1) with \( l = t \), one gets

\[
X_1^t F(f_1 V - \Phi(f_1) U, \ldots, f_r V - \Phi(f_r) U) \equiv g \Psi(F(T)) \pmod{I_2(X, Y)},
\]
where \( g \) is the polynomial given in (3-1).
where
\[
g = \sum_{l=0}^{t} (-1)^l \frac{t!}{(t-l)!} X_1^{(t-l)s} Y_1 s V_{t-l} U^l \not\in I_2(X, Y)[X, Y, U, V].
\]

One concludes that
\[
F(f_1 V - \Phi(f_1) U, \ldots, f_r V - \Phi(f_r) U) \in I_2(X, Y)[X, Y, U, V] \text{ if and only if } \Psi(F(T)) \in I_2(X, Y)[X, Y, U, V] \cap k[X] = (0).
\]

This finishes the proof.

(3.3) Corollary. Notation as in the beginning of the section. Moreover, let \( n = 2 \). Then \( \mathcal{F}(\tilde{I}) \) is isomorphic to the homogeneous coordinate ring of the duple Veronese model of \( \mathbb{P}^{n-1} \). In particular, \( \mathcal{F}(\tilde{I}) \) is normal and Cohen–Macaulay.

Proof. By Proposition (2.2.3), \( \mathcal{F}(\tilde{I}) \simeq k[M_\alpha] \), where \( M_\alpha \) runs through the monomials of degree \( d \) in the variables \( X \).

References

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