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ELEMENTARY OPERATORS AND INVARIANT SUBALGEBRAS

KEITH J. COATES

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Abstract. We provide an example of an elementary operator which leaves invariant a nest algebra but which cannot be written as a finite sum of multiplications each of which leaves the nest algebra invariant. We also prove that the given operator lies in the completely bounded norm closure of the linear span of the multiplications which leave the nest algebra invariant.

Let $\mathcal{A}$ be a Banach algebra and let $A, B \in \mathcal{A}$. A multiplication (operator) $A \cdot B : \mathcal{A} \to \mathcal{A}$ is defined by the equation $(A \cdot B)(X) = AXB, X \in \mathcal{A}$. An elementary operator R is an operator of the form $R = \sum_{i=1}^{n} A_i \cdot B_i$, where n is finite. Clearly, R can have different representations $\sum C_t \cdot D_t$. The length l of R is defined to be the smallest number of multiplication terms required for any representation of R.

Elementary operators have been studied in great detail in recent years. Both their spectral theory and their structural theory have received much attention (cf. [AF, Cu, F]). The range inclusion problem for elementary operators [F] can be stated as follows: If $\mathcal{I}$ is an ideal in $\mathcal{A}$, and if $Ran(R) \subseteq \mathcal{I}$, must R be expressible as $\sum_{i=1}^{n} A_i \cdot B_i$ with $Ran(A_i \cdot B_i) \subseteq \mathcal{I}, i = 1, 2, \ldots, n$? Positive results have been obtained with certain hypotheses [AF].

Let $\mathcal{C}$ be a subalgebra of $\mathcal{A}$. David Larson has suggested the study of $\mathcal{E}_{\mathcal{C}}(\mathcal{A})$, the algebra of elementary operators on $\mathcal{A}$ which leave $\mathcal{C}$ invariant. With $\mathcal{A} = B(l_2)$ and $\mathcal{C} = \mathcal{T}_\infty$, the nest algebra of upper triangular operators with respect to the standard basis in l_2, we provide in Theorem 3 an example of an $R \in \mathcal{E}_{\mathcal{C}}(\mathcal{A})$ which has no representation $\sum_{t=1}^{m} C_t \cdot D_t$ such that $C_t \cdot D_t \in \mathcal{E}_{\mathcal{C}}(\mathcal{A}),\ t = 1, 2, \ldots, m$. (This example was discussed, without proof, in [Co].) We also demonstrate in Proposition 4 that R lies in the completely bounded norm closed linear span of the length one elements of $\mathcal{E}_{\mathcal{C}}(\mathcal{A})$.

$\mathcal{T}_\infty$ is a commutative subspace lattice (CSL) algebra. Recall that a CSL algebra is a reflexive algebra whose lattice of invariant subspaces is commutative. For finite dimensional CSL algebras there is no operator like the operator R of the previous paragraph.

Proposition 1. *Let $\mathcal{C}$ be a CSL subalgebra of $\mathcal{M}_n$, and let $R \in \mathcal{E}_{\mathcal{C}}(\mathcal{M}_n)$ ($R \neq 0$). Then there exists a representation $R = \sum_{t=1}^{m} C_t \cdot D_t$ such that $C_t \cdot D_t \in \mathcal{E}_{\mathcal{C}}(\mathcal{M}_n),\ 1 \leq t \leq m$.*

Proof. Let $R = \sum_{s=1}^{p} A_s \cdot B_s$. Since $\mathcal{C}$ is a CSL algebra, it admits a star diagram, that is, there exists a subset $I \times J \subseteq \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ such that $\mathcal{C} = \{(a_{ij}) : a_{ij} \in C, (i, j) \in I \times J\}$ [KL].

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Let $A_1 = (a_{ij}) = \sum a_{ij}E_{ij},\ B_1 = (b_{ij}) = \sum b_{ij}E_{ij}$. With this, we see that we can expand $A_1 \cdot B_1$ as a linear combination of multiplications involving only pairs of matrix units: $A_1 \cdot B_1 = \sum_{i,j,k,l} \alpha_{ijkl}(E_{ij} \cdot E_{kl})$. Expanding $A_2 \cdot B_2, \ldots, A_p \cdot B_p$ in like manner and combining like terms, we arrive at

$$R = \sum \beta_{ijkl}(E_{ij} \cdot E_{kl}) \tag{1}$$

Let $E_{ab} \in \mathcal{C}$ and assume that $R(E_{ab}) \neq 0$. Then

$$R(E_{ab}) = \sum_{i,j} \beta_{iabj}(E_{ia} \cdot E_{bj})E_{ab} = \sum_{i,j} \beta_{iabj}E_{ij}$$

and there is a (c,d) such that $\beta_{cabd} \neq 0$. Then $E_{cd} \in \mathcal{C}$, for if not, we would have that $\sum \beta_{iabj}E_{ij} \notin \mathcal{C}$, contradicting our hypothesis. So every nonzero multiplication $\beta_{iabj}(E_{ia} \cdot E_{bj})$ maps E_{ab} into $\mathcal{C}$ and every multiplication in (1) either annihilates E_{ab} or maps it onto a nonzero element of $\mathcal{C}$. Thus $\sum \beta_{ijkl}(E_{ij} \cdot E_{kl})$ is an expansion of R such that each multiplication lies in $\mathcal{E}_{\mathcal{C}}(\mathcal{M}_n)$. □

Note: The conclusion of Proposition 1 can be false if $\mathcal{C}$ is not a CSL subalgebra of $\mathcal{M}_n$ (see Example 5).

Clearly, the number of terms in (1) can be arbitrarily large for arbitrarily large n. The observation of this fact in a specific instance will facilitate the proof of Theorem 3.

Lemma 2. *Let $n > 2$, and let $\mathcal{T}_n \subseteq \mathcal{M}_n$ be the algebra of upper triangular $n \times n$ matrices. There exists an $R \in \mathcal{E}_{\mathcal{T}_n}(\mathcal{M}_n)$ of length two such that for any representation $R = \sum_{t=1}^{m} C_t \cdot D_t$ with each $C_t \cdot D_t \in \mathcal{E}_{\mathcal{T}_n}(\mathcal{M}_n)$, we must have $m \geq \log_2 n$.*

Proof. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ be a sequence of distinct complex numbers and define the following four matrices in $\mathcal{M}_n$:

$$A_1 = \sum_{i=2}^{n} E_{ii}, \qquad B_1 = \sum_{j=2}^{n} \alpha_{j-1}E_{j(j-1)},$$

$$A_2 = \sum_{k=2}^{n} \alpha_{k-1}E_{kk}, \qquad B_2 = \sum_{l=2}^{n} -E_{l(l-1)}.$$

Consider the elementary operator $R = A_1 \cdot B_1 + A_2 \cdot B_2$. By Proposition 1, R expands as

$$\sum_{i=2}^{n}\sum_{j=2}^{n} (\alpha_{j-1} - \alpha_{i-1})(E_{ii} \cdot E_{j(j-1)});$$

$R \in \mathcal{E}_{\mathcal{T}_n}(\mathcal{M}_n)$ since it takes strictly upper triangular matrix units to upper triangular matrix units, and the diagonal matrix units are annihilated.

Since the α_k's are all distinct, any expansion of R must include nonzero multiples of every one of the terms

$$E_{ii} \cdot E_{j(j-1)}$$

for $i \neq j, 2 \leq i,j \leq n$. On the other hand, if $C \cdot D \in \mathcal{E}_{\mathcal{T}_n}(\mathcal{M}_n)$ is to contain a nonzero multiple of $E_{ii} \cdot E_{j(j-1)}$ then C must contain a nonzero multiple of E_{ii} and D must contain a nonzero multiple of $E_{j(j-1)}$. Thus, if $C \cdot D$ contains nonzero multiples of $E_{ii} \cdot E_{j(j-1)}$ and $E_{kk} \cdot E_{l(l-1)}$ then it must also contain a nonzero multiple of $E_{kk} \cdot E_{j(j-1)}$. If $C \cdot D$ is to leave $\mathcal{T}_n$ invariant, we must have $k \neq j$.

Now suppose that $R = \sum_{t=1}^{m} C_t \cdot D_t$. For each i let W_i be the set of t's for which $C_t \cdot D_t$ contains a nonzero multiple of $E_{ii} \cdot E_{k(k-1)}$. We claim the sets W_i are all distinct. If $i \neq j$, the last paragraph implies that there is a $t_0 \in W_i$ such that $C_{t_0} \cdot D_{t_0}$ contains a nonzero multiple of $E_{ii} \cdot E_{j(j-1)}$. But, also by the last paragraph, this means that $C_{t_0} \cdot D_{t_0}$ cannot contain nonzero multiples of *any* operator of the form $E_{jj} \cdot E_{l(l-1)}$. This tells us that $t_0 \notin W_j$, so $W_j \neq W_i$.

Thus, the total number of sets W_i (that is, $n-1$) is no greater than the total number of distinct nonempty subsets of $\{1, 2, \dots, m\}$, which is $2^m - 1$. The result follows. □

Theorem 3. *There exists an elementary operator $R \in \mathcal{E}_{\mathcal{T}_\infty}\big(B(l_2)\big)$ of length two which cannot be written as a finite sum $\sum_{t=1}^{m} C_t \cdot D_t$ of multiplications with $C_t \cdot D_t \in \mathcal{E}_{\mathcal{T}_\infty}\big(B(l_2)\big)$ for $1 \leq t \leq m$.*

Proof. Let $\alpha = (\alpha_1, \alpha_2, \dots)$ be a sequence in l_∞ with the α_i's all distinct. Form the following four operators in $B(l_2)$:

$$A_1 = \sum_{i=2}^{\infty} E_{ii}, \qquad B_1 = \sum_{j=2}^{\infty} \alpha_{j-1} E_{j(j-1)},$$

$$A_2 = \sum_{k=2}^{\infty} \alpha_{k-1} E_{kk}, \qquad B_2 = \sum_{l=2}^{\infty} -E_{l(l-1)}.$$

Let $R = A_1 \cdot B_1 + A_2 \cdot B_2$. Again, routine computations show that $R \in \mathcal{E}_{\mathcal{T}_\infty}\big(B(l_2)\big)$, but $A_1 \cdot B_1 \notin \mathcal{E}_{\mathcal{T}_\infty}\big(B(l_2)\big)$, $A_2 \cdot B_2 \notin \mathcal{E}_{\mathcal{T}_\infty}\big(B(l_2)\big)$.

Suppose we can write R as a finite sum $\sum_{t=1}^{m} C_t \cdot D_t$ with $C_t \cdot D_t \in \mathcal{E}_{\mathcal{T}_\infty}\big(B(l_2)\big)$, $1 \leq t \leq m$. With n fixed and P_n the projection onto the first n basis vectors, let $X' = P_n X P_n \in \mathcal{M}_n$ for $X \in B(l_2)$. Let $R' = A_1' \cdot B_1' + A_2' \cdot B_2'$. Then R' is an elementary operator of the type constructed in Lemma 2. Further, $R' = \sum_{t=1}^{m} C_t' \cdot D_t'$ with $C_t' \cdot D_t' \in \mathcal{E}_{\mathcal{T}_n}(\mathcal{M}_n)$ for $1 \leq t \leq m$. Lemma 2 therefore implies that $m \geq \log_2 n$. But this must be true for all $n > 2$ with m fixed, an impossibility.

We conclude that R cannot be written as a finite sum $\sum_{t=1}^{m} C_t \cdot D_t$ with $C_t \cdot D_t \in \mathcal{E}_{\mathcal{T}_\infty}\big(B(l_2)\big)$. □

If $\mathcal{A}$ is a C*-algebra and $\phi : \mathcal{A} \to \mathcal{A}$ is a bounded linear operator, define $\phi_n : M_n(\mathcal{A}) \to M_n(\mathcal{A})$ by the equation $\phi_n\big((a_{ij})\big) = \big(\phi(a_{ij})\big)_{ij}$. If $sup_n \|\phi_n\| < \infty$, say that ϕ is completely bounded and define the completely bounded norm $\|\phi\|_{cb}$ of ϕ to be this supremum. $CB(\mathcal{A})$, the algebra of completely bounded operators, contains the elementary operators as a proper (nonclosed) subalgebra.

If $\mathcal{A} \odot \mathcal{B}$ is the algebraic tensor product of two operator algebras $\mathcal{A}, \mathcal{B}$, define $\|u\|_h$, the Haagerup norm of $u \in \mathcal{A} \odot \mathcal{B}$, by

$$\|u\|_h = inf\Big\{ \|(a_1, a_2, \dots, a_n)\| \|(b_1, b_2, \dots, b_n)^t\| : u = \sum_{i=1}^{n} a_i \otimes b_i \Big\}.$$

The closure of $\mathcal{A} \odot \mathcal{B}$ in the topology defined by this norm is known as the Haagerup tensor product of $\mathcal{A}$ and $\mathcal{B}$ and is denoted by $\mathcal{A} \otimes_h \mathcal{B}$. For an excellent overview of the theory of completely bounded operators and the Haagerup tensor product, see [ChrSi].

There are rich connections between the structures of $CB(\mathcal{A})$ and of $\mathcal{A} \odot \mathcal{A}$. In [CSi] it is shown that the map $\theta : \mathcal{A} \odot \mathcal{A} \to CB(\mathcal{A})$, given by $\theta(a \otimes b)(x) = axb, x \in \mathcal{A}$, is an isometry.

Proposition 4. *Let $\mathcal{F}$ be the linear span of the length one elements of $\mathcal{E}_{\mathcal{T}_\infty}\big(B(l_2)\big)$, and let $\overline{\mathcal{F}}$ be the closure of $\mathcal{F}$ in the completely bounded norm topology. Then $R \in \overline{\mathcal{F}}$, where R is the operator constructed in Theorem 3.*

Proof. Without loss of generality, assume $\|\alpha\|_\infty \leq 1$. Let ϵ be given, and let $\beta = (\beta_1', \beta_2', \dots) \in l_\infty$ be a simple (i.e., finitely valued) $\epsilon/3$ approximation of α such that $\|\beta\|_\infty \leq 1$. Let $\{\beta_1, \beta_2, \dots, \beta_m\}$ be the set of distinct values which β assumes. Form the following four operators in $B(l_2)$:

$$C_1 = \sum_{i=2}^{\infty} E_{ii}, \qquad D_1 = \sum_{j=2}^{\infty} \beta_{j-1}' E_{j(j-1)},$$

$$C_2 = \sum_{k=2}^{\infty} \beta_{k-1}' E_{kk}, \qquad D_2 = \sum_{l=2}^{\infty} -E_{l(l-1)},$$

and let $S = C_1 \cdot D_1 + C_2 \cdot D_2$. We first show that $S \in \mathcal{F}$. Let $Q_1, Q_2, \dots, Q_m$ be diagonal projections such that $\sum_{i=1}^m \beta_i Q_i = C_2$. Let $T_1, T_2, \dots, T_m$ be operators with only ones and zeroes on the first subdiagonal and zeroes elsewhere such that $\sum_{i=1}^m \beta_i T_i = D_1$.

Now, for $E_{ij} \in \mathcal{T}_\infty, E_{ii}(E_{ij})E_{j(j-1)} = E_{i(j-1)} \in \mathcal{T}_\infty$ as long as $i \neq j$ (for then $i < j$, so that $i \leq j-1$). Thus, $Q_r \cdot T_s \in \mathcal{E}_{\mathcal{T}_\infty}\big(B(l_2)\big)$ for $r \neq s$. We have

$$\begin{aligned} C_1 \cdot D_1 + C_2 \cdot D_2 &= \left(\left(\sum Q_r\right) \cdot \left(\sum \beta_s T_s\right)\right) - \left(\left(\sum \beta_r Q_r\right) \cdot \left(\sum T_s\right)\right) \\ &= \sum_{1 \leq r,s \leq m} \beta_s (Q_r \cdot T_s) - \sum_{1 \leq r,s \leq m} \beta_r (Q_r \cdot T_s) = \sum_{r \neq s} \beta_s (Q_r \cdot T_s) - \beta_r (Q_r \cdot T_s). \end{aligned}$$

Each term in the above (finite) sum lies in $\mathcal{E}_{\mathcal{T}_\infty}\big(B(l_2)\big)$, so that $S = C_1 \cdot D_1 + C_2 \cdot D_2 \in \mathcal{F}$.

We now show that $\|S - R\|_{cb} < \epsilon$. We have that $\|A_1 - C_1\| = \|B_2 - D_2\| = 0$, and $\|A_2 - C_2\| < \epsilon/3, \|B_1 - D_1\| < \epsilon/3$. Consider the following operator vectors:

$$\left(0, \sqrt{\frac{\epsilon}{3}} C_1, \sqrt{\frac{3}{\epsilon}} (A_2 - C_2), \sqrt{\frac{\epsilon}{3}} C_2\right), \quad \left(\sqrt{\frac{\epsilon}{3}} B_1, \sqrt{\frac{3}{\epsilon}} (B_1 - D_1), \sqrt{\frac{\epsilon}{3}} B_2, 0\right)^t.$$

By construction, each has norm less than $\sqrt{\epsilon}$. Thus, ϵ is larger than

$$\begin{aligned} &\left\| 0 \otimes \sqrt{\frac{\epsilon}{3}} B_1 + \sqrt{\frac{\epsilon}{3}} C_1 \otimes \sqrt{\frac{3}{\epsilon}} (B_1 - D_1) + \sqrt{\frac{3}{\epsilon}} (A_2 - C_2) \otimes \sqrt{\frac{\epsilon}{3}} B_2 + \sqrt{\frac{\epsilon}{3}} C_2 \otimes 0 \right\|_h \\ &= \big\| (A_1 - C_1) \otimes B_1 + C_1 \otimes (B_1 - D_1) + (A_2 - C_2) \otimes B_2 + C_2 \otimes (B_2 - D_2) \big\|_h \\ &= \big\| \big[(A_1 \otimes B_1) + (A_2 \otimes B_2)\big] - \big[(C_1 \otimes D_1) + (C_2 \otimes D_2)\big] \big\|_h. \end{aligned}$$

Theorem 4.3 in [CSi] now implies that $\|S - R\|_{cb} < \epsilon$.

Since ϵ was arbitrary, $R \in \overline{\mathcal{F}}$. $\square$

We close with a finite dimensional example, which demonstrates that if $\mathcal{C}$ is not a CSL subalgebra of $\mathcal{M}_n$, then the conclusion of proposition 1 can be false.

Example 5. Let $\mathcal{A} = \mathcal{M}_2$, the algebra of 2×2 matrices over C. Let $\mathcal{C}$ be the triangular Toeplitz subalgebra

$$\mathcal{C} = \left\{ \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix} : \lambda, \mu \in \mathbf{C} \right\}.$$

Let

$$A_1 = B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},$$

and let $R = A_1 \cdot B_1 + A_2 \cdot B_2$. It is not difficult to verify that

$$R \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & d-a \end{pmatrix} \qquad (\text{so } R \neq 0),$$

and

$$R \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda - \lambda \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, $R \in \mathcal{E}_{\mathcal{C}}(\mathcal{A})$ (in fact, R annihilates $\mathcal{C}$).

Routine computations now show that R does not lie in (the norm closure of) the linear span of length one elements of $\mathcal{E}_{\mathcal{C}}(\mathcal{A})$.

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Department of Mathematics, Illinois Wesleyan University, Bloomington, Illinois 61702

E-mail address: `kcoates@sun.iwu.edu`