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A CLOSED MODEL CATEGORY FOR $(n-1)$-CONNECTED SPACES

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Abstract. For each integer $n > 0$, we give a distinct closed model category structure to the category of pointed spaces $\mathrm{Top}_\star$ such that the corresponding localized category $\mathrm{Ho}(\mathrm{Top}_\star^n)$ is equivalent to the standard homotopy category of $(n-1)$-connected CW-complexes.

The structure of closed model category given by Quillen to $\mathrm{Top}_\star$ is based on maps which induce isomorphisms on all homotopy group functors π_q and for any choice of base point. For each $n > 0$, the closed model category structure given here takes as weak equivalences those maps that for the given base point induce isomorphisms on π_q for $q \geq n$.

§0. Introduction

D. Quillen [7] introduced the notion of closed model category and proved that the categories of spaces and of simplicial sets have the structure of a closed model category. This structure has been very useful in the development of the homotopy theory. For example, Quillen [8] used this structure to find algebraic models for rational homotopy theory.

Recently, C. Elvira and L.J. Hernández [2] have given a closed model structure for the notion of n-type introduced by Whitehead. They take as weak equivalences those maps $f\colon X \longrightarrow Y$ which induce isomorphisms on the homotopy functors π_q for $q \leq n$. In this case, the localized category $\mathrm{Ho}_n(\mathrm{Top})$ obtained by formal inversion of the corresponding weak equivalences is equivalent to $\mathrm{Ho}(\mathrm{Top})|_{(n+1)\text{-coconnected}}$, the full subcategory of the localized category $\mathrm{Ho}(\mathrm{Top})$ determined by the $(n+1)$-coconnected topological spaces ($\pi_q = 0, q \geq n+1$).

The aim of this paper is to study the homotopy category of $(n-1)$-connected spaces, which in some sense is dual to the homotopy category of n-coconnected spaces. Quillen [8] writes: "It is unfortunate that the category Top_n of $(n-1)$-connected topological spaces is not closed under finite limits, for this prevents us from making this category into a closed model category." However, he solves the problem by considering the category of n-reduced simplicial sets SS_n and the functor $E_n\,\mathrm{Sing}$ that induces an equivalence between the localized categories $\mathrm{Ho}(\mathrm{Top}_n)$,

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$\mathrm{Ho}(\mathrm{SS}_n)$. When some homotopy construction needs finite limits, then the functor E_n Sing carries the corresponding diagrams to the category $\mathrm{Ho}(\mathrm{SS}_n)$ which is closed under finite limits, and afterwards the realization functor carries again the diagrams to the category of $(n-1)$-connected spaces. We propose a different solution; we use the category of pointed spaces which is closed under finite limits. As Quillen we also have equivalent localized categories, but now all the usual homotopy constructions can be done in the category of pointed spaces.

In this paper, for each $n > 0$, we take as weak n-equivalences those maps of $\mathrm{Top}_\star$ which induce isomorphisms on the homotopy group functors π_q for $q \geq n$. We complete this family of weak n-equivalences with families of n-fibrations and n-cofibrations in such a way that $\mathrm{Top}_\star$ admits the structure of a closed model category and its localized category is equivalent to the localized category of $(n-1)$-connected spaces and to the localized category of n-reduced simplicial sets.

§1. Definitions and statement of the Theorem

We begin by recalling the definition of a closed model category given by Quillen [8].

Definition 1.1. A closed model category $\mathcal{C}$ is a category endowed with three distinguished families of maps called cofibrations, fibrations and weak equivalences satisfying the axioms $CM1$–$CM5$ below:

$CM1$. $\mathcal{C}$ is closed under finite projective and inductive limits.

$CM2$. If f and g are maps such that gf is defined, then, whenever two of these f, g and gf are weak equivalences, so is the third.

Recall that the maps in $\mathcal{C}$ form the objects of a category $Maps(\mathcal{C})$ having commutative squares for morphisms. We say that a map f in $\mathcal{C}$ is a retract of g if there are morphisms $\varphi\colon f \longrightarrow g$ and $\psi\colon g \longrightarrow f$ in $Maps(\mathcal{C})$ such that $\psi\varphi = id_f$.

A map which is a weak equivalence and a fibration is said to be a trivial fibration, and, similarly, a map which is a weak equivalence and a cofibration is said to be a trivial cofibration.

$CM3$. If f is a retract of g and g is a fibration, cofibration or weak equivalence, then so is f.

$CM4$. (Lifting.) Given a solid arrow diagram

$$(*) \qquad \begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

the dotted arrow exists in either of the following situations:

(i) i is a cofibration and p is a trivial fibration,
(ii) i is a trivial cofibration and p is a fibration.

$CM5$. (Factorization.) Any map f may be factored in two ways:
(i) $f = pi$, where i is a cofibration and p is a trivial fibration,
(ii) $f = qj$, where j is a trivial cofibration and q is a fibration.

We say that a map $i\colon A \longrightarrow B$ in a category has the left lifting property (LLP) with respect to another map $p\colon X \longrightarrow Y$, and p is said to have the right lifting property (RLP) with respect to i if the dotted arrow exists in any diagram of the form $(*)$.

The initial object of $\mathcal{C}$ is denoted by $\emptyset$ and the final object by $\star$. An object X of $\mathcal{C}$ is said to be fibrant if the morphism $X \longrightarrow \star$ is a fibration and it is said cofibrant if $\emptyset \longrightarrow X$ is a cofibration.

We shall use the following notation: For each integer $n \geq 0$, $\triangle[n]$ denotes the "standard n-simplex", and for $n > 0$, $\dot{\triangle}[n]$ (resp. $\mathrm{V(n,k)}$ for $0 \leq k \leq n$) denotes the simplicial subset of $\triangle[n]$ which is the union of the images of the faces $\partial_i\colon \triangle[n-1] \longrightarrow \triangle[n]$ for $0 \leq i \leq n$ (resp. $0 \leq i \leq n$, $i \neq k$.) We write $\mathrm{sk}_q(\ )$ for the q-skeleton functor and $|\ |$ for the geometric realization functor.

In this paper, the following closed model categories given by Quillen [7], [8] will be considered:

(1) The category of pointed topological spaces $\mathrm{Top}_\star$: Given a map $f\colon X \longrightarrow Y$ in $\mathrm{Top}_\star$, f is said to be a fibration if it is a fibre map in the sense of Serre; f is a weak equivalence if f induces isomorphisms $\pi_q(f)$ for $q \geq 0$ and for any choice of base point, and f is a cofibration if it has the LLP with respect to all trivial fibrations.

(2) The category of pointed simplicial sets $\mathrm{SS}_\star$: A map $f\colon X \longrightarrow Y$ in $\mathrm{SS}_\star$ is said to be a fibration if f is a fibre map in the sense of Kan; f is a weak equivalence if its geometric realization, $|f|$, is a homotopy equivalence, and f is a cofibration if it has the LLP with respect to any trivial fibration.

(3) The category of n-reduced simplicial sets SS_n . A pointed simplicial set X is said to be n-reduced if $\mathrm{sk}_{n-1} X$ is isomorphic to the simplicial subset generated by the base 0-simplex of X . We write SS_n for the full subcategory of $\mathrm{SS}_\star$ determined by all the n-reduced simplicial sets. A map $f\colon X \longrightarrow Y$ in SS_n is said to be a cofibration in SS_n if f is injective, f is a weak equivalence if it is a weak equivalence in $\mathrm{SS}_\star$, and f is a fibration if it has the RLP with respect to the trivial cofibrations in SS_n .

Let $\mathrm{Ho}(\mathrm{Top}_\star)$, $\mathrm{Ho}(\mathrm{SS}_\star)$, and $\mathrm{Ho}(\mathrm{SS}_n)$ denote the corresponding localized categories obtained by formal inversion of the respective families of weak equivalences defined above.

In the category of pointed topological spaces and continuous maps, $\mathrm{Top}_\star$, for each integer $n > 0$, we consider the following families of maps:

Definition 1.2. Let $f : X \longrightarrow Y$ be a map in $\mathrm{Top}_\star$.

(i) f is a weak n-equivalence if the induced map $\pi_q(f)\colon \pi_q(X) \longrightarrow \pi_q(Y)$ is an isomorphism for each $q \geq n$.

(ii) f is an n-fibration if it has the RLP with respect to the inclusions

$$|\,\mathrm{V(p,k)}\,/\,\mathrm{sk}_{n-1}\,\mathrm{V(p,k)}\,| \longrightarrow |\,\triangle[p]\,/\,\mathrm{sk}_{n-1}\,\triangle[p]\,|$$

for every $p > n$ and $0 \leq k \leq p$.

A map which is both an n-fibration and a weak n-equivalence is said to be a trivial n-fibration.

(iii) f is an n-cofibration if it has the LLP with respect to any trivial n-fibration.

A map which is both an n-cofibration and a weak n-equivalence is said to be a trivial n-cofibration.

A pointed space X is said to be n-fibrant if the map $X \longrightarrow \star$ is an n-fibration, and X is said to be n-cofibrant if the map $\star \longrightarrow X$ is an n-cofibration.

Remark. We note that the homotopy group $\pi_q(X)$ only depends on the path component C of the given base point of X . Therefore the inclusion $C \longrightarrow X$ is always a weak n-equivalence. On the other hand, the objects $|\,\mathrm{V(p,k)}\,/\,\mathrm{sk}_{n-1}\,\mathrm{V(p,k)}\,|$, $|\,\triangle[p]\,/\,\mathrm{sk}_{n-1}\,\triangle[p]\,|$ used in the definition of n-fibration are considered as pointed spaces. It is also clear that all objects in $\mathrm{Top}_\star$ are n-fibrant.

Using the notions given above the main result of this paper is:

Theorem 1.3. *For each $n > 0$, the category $\mathrm{Top}_\star$ together with the families of n-fibrations, n-cofibrations and weak n-equivalences, has the structure of a closed model category.*

We denote by $\mathrm{Top}_\star^n$ the closed model category $\mathrm{Top}_\star$ with the distinguised families of n-fibrations, n-cofibrations and weak n-equivalences, and by $\mathrm{Ho}(\mathrm{Top}_\star^n)$ the category of fractions obtained from $\mathrm{Top}_\star^n$ by formal inversion of the family of weak n-equivalences.

§2. Proof of the Theorem

It is well known that Axiom $CM1$ is satisfied by $\mathrm{Top}_\star$, Axiom $CM2$ is an immediate consequence of the properties of group isomorphisms, and the definition of n-cofibration implies obviously Axiom $CM4$ (i). Then, it only remains to prove Axioms $CM3$, $CM4$ (ii) and $CM5$.

Theorem 1.3 will follow from the results given below.

Lemma 2.1. *If a map f is a retract of a map g and g has the RLP (resp. LLP) with respect to another map h, then f also has this property.*

Proposition 2.2 (Axiom $CM3$)**.** *In $\mathrm{Top}_\star$ if a map f is a retract of a map g and g is an n-fibration, n-cofibration or weak n-equivalence, then so is f.*

Proof. If g is an n-fibration or an n-cofibration, by Lemma 2.1 we have the same for f. If g is a weak n-equivalence, since $\pi_q f$ is a retract of $\pi_q g$ it follows that f is also a weak n-equivalence.

Proposition 2.3. *For a map $f\colon X \longrightarrow Y$ in $\mathrm{Top}_\star$, the following statements are equivalent:*

(i) *f is a trivial n-fibration,*
(ii) *f has the RLP with respect to the inclusions*

$$|\,\dot{\triangle}[p]\,/\,\mathrm{sk}_{n-1}\,\dot{\triangle}[p]\,| \longrightarrow |\,\triangle[p]\,/\,\mathrm{sk}_{n-1}\,\triangle[p]\,|$$

for all integers $p \geq n$.

Proof. Let $\mathrm{Sing}\colon \mathrm{Top}_\star \longrightarrow \mathrm{SS}_\star$ denote the "singular" functor which is right adjoint to the "realization" functor $|\ |\colon \mathrm{SS}_\star \longrightarrow \mathrm{Top}_\star$. Consider also the "n-reduction" functor $R_n\colon \mathrm{SS}_\star \longrightarrow \mathrm{SS}_\star$ defined as follows: Given a pointed simplicial set X, the n-reduction $R_n(X)$ is the simplicial subset of X of those simplices of X whose q-faces for $q < n$ are degeneracies of the base 0-simplex. The left adjoint of R_n is the functor $(\ )_{(n)}\colon \mathrm{SS}_\star \longrightarrow \mathrm{SS}_\star$ defined by $X_{(n)} = X/\,\mathrm{sk}_{n-1}\,X$. Then the composite functor $|\ |(\ )_{(n)}$ is left adjoint to the functor $R_n\,\mathrm{Sing}$.

On the other hand, because for any pointed space X, $\mathrm{Sing}\,X$ is a Kan simplicial set, $R_n\,\mathrm{Sing}\,X$ is the n-Eilenberg subcomplex of $\mathrm{Sing}\,X$. Therefore, for $q \geq n$, we have the isomorphisms

$$\pi_q(R_n\,\mathrm{Sing}\,X) \cong \pi_q(\mathrm{Sing}\,X) \cong \pi_q(X).$$

Taking into account the above remarks, for a map $f\colon X \longrightarrow Y$ in $\mathrm{Top}_\star$ we have that f is a trivial n-fibration if and only if $R_n \operatorname{Sing} f$ has the RLP with respect to the inclusions $\mathrm{V(p,k)} \longrightarrow \triangle[p],\ p > n, 0 \le k \le p$, and $\pi_q(R_n \operatorname{Sing} f)$ is an isomorphism for $q \ge 0$. By [8, Proposition 2.12] the last conditions are equivalent to the fact that $R_n \operatorname{Sing} f$ is a trivial fibration in SS_n . Now we can apply [8, Proposition 2.3] and, in this case, this is equivalent to saying that $R_n \operatorname{Sing} f$ is a trivial fibration of $\mathrm{SS}_\star$. However, the trivial fibrations in $\mathrm{SS}_\star$ are characterized by the RLP with respect to $\dot{\triangle}[p] \longrightarrow \triangle[p],\ p > 0$.

Applying again that $R_n \operatorname{Sing}$ and $|\,|(\ )_{(n)}$ are adjoint functors, we conclude that f is a trivial n-fibration if and only if f has the RLP with respect to

$$|\,\dot{\triangle}[p]\,/\,\mathrm{sk}_{n-1}\,\dot{\triangle}[p]\,| \longrightarrow |\,\triangle[p]\,/\,\mathrm{sk}_{n-1}\,\triangle[p]\,|,\ p \ge n.$$

Proposition 2.4 (Axiom $CM5$)**.** *Let $f\colon X \longrightarrow Y$ be a map in $\mathrm{Top}_\star$; then f can be factored in two ways:*

(i) *$f = pi$, where i is an n-cofibration and p is a trivial n-fibration,*
(ii) *$f = qj$, where j is a weak n-equivalence having the LLP with respect to all n-fibrations and q is an n-fibration.*

Proof. Given a class $\mathcal{M}$ of maps, denote by $\mathcal{M}'$ the class of maps which have the RLP with respect to the maps of $\mathcal{M}$.

(i) Consider the family $\mathcal{M}$ of inclusion maps

$$|\,\dot{\triangle}[r]\,/\,\mathrm{sk}_{n-1}\,\dot{\triangle}[r]\,| \longrightarrow |\,\triangle[r]\,/\,\mathrm{sk}_{n-1}\,\triangle[r]\,|,\ r \ge n.$$

By Proposition 2.3, $\mathcal{M}'$ is the class of trivial n-fibrations.

At this point, in order to use the "small object argument", we refer the reader to Lemma 3 of Chapter II, §3 of Quillen [7]. In a similar way, we construct a diagram

$$\begin{tikzcd} X \arrow[r, "f_0"] \arrow[dr, "f"'] & Z^0 \arrow[r, "f_1"] \arrow[d, "p^0"] & Z^1 \arrow[r] \arrow[dl, "p^1"] & \cdots \\ & Y & & \end{tikzcd}$$

but in our case, we use all commutative diagrams of the form

$$\begin{tikzcd} |\,\dot{\triangle}[r]\,/\,\mathrm{sk}_{n-1}\,\dot{\triangle}[r]\,| \arrow[r] \arrow[d] & Z^{k-1} \arrow[d, "p^{k-1}"] \\ |\,\triangle[r]\,/\,\mathrm{sk}_{n-1}\,\triangle[r]\,| \arrow[r] & Y \end{tikzcd}$$

for every $r \ge n$. We also consider $Z = \operatorname{colim} Z^k$, $p = \operatorname{colim} p^k$ and $i\colon X \longrightarrow Z$ the induced inclusion. With this construction, we can factor $f\colon X \longrightarrow Y$ as $f = pi$, where $p\colon Z \longrightarrow Y$ is in $\mathcal{M}'$ and $i\colon X \longrightarrow Z$ has the LLP with respect to the maps of $\mathcal{M}'$. Then, p is a trivial n-fibration and i is an n-cofibration. It is interesting to note that this construction factors maps f in a functorial way.

(ii) Now, take the family $\mathcal{M}$ of maps

$$|\,\mathrm{V(r,k)}\,/\,\mathrm{sk}_{n-1}\,\mathrm{V(r,k)}\,| \longrightarrow |\,\triangle[r]\,/\,\mathrm{sk}_{n-1}\,\triangle[r]\,|,\ r > n, 0 \le k \le r.$$

By Definition 1.2, $\mathcal{M}'$ is the class of n-fibrations. Analogously to (i), we can factor $f = qj$, where q is an n-fibration and j has the LLP with respect to all n-fibrations.

To complete the proof, we note that the inclusion

$$|\,\mathrm{V(r,k)}\,/\,\mathrm{sk}_{n-1}\,\mathrm{V(r,k)}\,| \longrightarrow |\,\triangle[r]\,/\,\mathrm{sk}_{n-1}\,\triangle[r]\,|,\ \ r > n, 0 \leq k \leq r,$$

is a trivial cofibration in Top . Therefore, for any map $|\,\mathrm{V(r,k)}\,/\,\mathrm{sk}_{n-1}\,\mathrm{V(r,k)}\,| \longrightarrow X$ in $\mathrm{Top}_\star$, $r > n, 0 \leq k \leq r$, the inclusion

$$X \longrightarrow X \bigcup_{|\,\mathrm{V(r,k)}\,/\,\mathrm{sk}_{n-1}\,\mathrm{V(r,k)}\,|} |\,\triangle[r]\,/\,\mathrm{sk}_{n-1}\,\triangle[r]\,|$$

is a trivial cofibration. Using this fact we can check that the map j is a weak n-equivalence.

Remark. Note that if $X = \star$, the n-cofibrant space Z constructed in the proof of (i) is $(n-1)$-connected. In this case, we denote Z by Y^n. This construction induces a well defined functor $\mathrm{Top}_\star \longrightarrow \mathrm{Top}_\star$, $Y \longrightarrow Y^n$.

Proposition 2.5 (Axiom $CM4$ (ii)). *Given a commutative diagram in* $\mathrm{Top}_\star$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

where i is a trivial n-cofibration and p is an n-fibration, then there is a lifting.

Proof. From Proposition 2.4 (ii), i can be factored as $i = qj$, where $j\colon A \longrightarrow W$ is a weak n-equivalence having the LLP with respect to all n-fibrations and $q\colon W \longrightarrow Y$ is an n-fibration. Since $CM2$ is verified, q is a trivial n-fibration. Then, there is a lifting $r\colon B \longrightarrow W$ for the commutative diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & W \arrow[d, "q"] \\ B \arrow[r, "id"'] & B \end{tikzcd}$$

So, the map i is a retract of j and applying Lemma 2.1, it follows that a lifting exists for the given diagram.

§3. The category $\mathrm{Ho}(\mathrm{Top}^n_\star)$

In this section, we compare the closed model category $\mathrm{Top}^n_\star$ with the closed model categories $\mathrm{Top}_\star$ and SS_n (see §1). We obtain that the localized category $\mathrm{Ho}(\mathrm{Top}^n_\star)$ is equivalent to $\mathrm{Ho}(\mathrm{SS}_n)$ and to $\mathrm{Ho}(\mathrm{Top}_\star)|_{(n-1)\text{-connected}}$, the full subcategory of $\mathrm{Ho}(\mathrm{Top}_\star)$ determined by $(n-1)$-connected spaces.

We note that the identity functor $Id\colon \mathrm{Top}_\star \longrightarrow \mathrm{Top}^n_\star$ preserves weak equivalences and fibrations. One also has the following result:

Proposition 3.1. (i) *If Y is n-cofibrant, then Y is $(n-1)$-connected.*

(ii) *If $f\colon X \longrightarrow Y$ is a weak n-equivalence and X, Y are n-cofibrant, then f is a weak equivalence.*

Proof. (i) Let $j\colon \star \longrightarrow Y$ be an n-cofibration. Applying Proposition 2.4 (i) and the remark after it, j can be factored as $j = pi\colon\ \star \xrightarrow{\quad} Y^n \xrightarrow{\quad} Y$, where i is an n-cofibration, p is a trivial n-fibration and Y^n is an $(n-1)$-connected space.

Because p is a trivial n-fibration and j is an n-cofibration, j is a retract of i. It follows that $\pi_q(j)$ is an isomorphism for $q \leq n-1$, hence Y is $(n-1)$-connected.

(ii) For $q \geq n$, we have that $\pi_q(f)$ is an isomorphism. On the other hand, since X, Y are n-cofibrant, one can apply (i) to obtain that X, Y are $(n-1)$connected, hence for $0 \leq q < n$ we also have that $\pi_q(f)$ is an isomorphism. Since $n > 0$, it follows that f is a weak equivalence.

Recall that if $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a functor between closed model categories, and F carries a weak equivalence between cofibrant objects of $\mathcal{C}$ into a weak equivalence of $\mathcal{D}$, there exists a left derived functor $F^L\colon \operatorname{Ho}(\mathcal{C}) \longrightarrow \operatorname{Ho}(\mathcal{D})$ defined by $F^L(X) = F(LX)$, where $LX \longrightarrow X$ is a trivial fibration and LX is a cofibrant object in $\mathcal{C}$. In a dual context one has right derived functors.

For example, by Proposition 3.1 the identity functor $Id\colon \mathrm{Top}^n_\star \longrightarrow \mathrm{Top}_\star$ carries weak n-equivalences between n-cofibrant spaces to weak equivalences, therefore one has the left derived functor $Id^L\colon \operatorname{Ho}(\mathrm{Top}^n_\star) \longrightarrow \operatorname{Ho}(\mathrm{Top}_\star)$ defined by $Id^L(X) = X^n$, where $X^n \longrightarrow X$ is a trivial n-fibration and X^n is an n-cofibrant space, which can be obtained by the functorial factorization given in Proposition 2.4 (i).

By the properties of the adjunction

$$\mathrm{Top}^n_\star \underset{Id}{\overset{Id}{\rightleftarrows}} \mathrm{Top}_\star$$

with respect to the families of cofibrations, fibrations and weak equivalences, we have the right derived functor $Id^R = Id\colon \operatorname{Ho}(\mathrm{Top}_\star) \longrightarrow \operatorname{Ho}(\mathrm{Top}^n_\star)$ and the left derived functor $Id^L = (\ )^n\colon \operatorname{Ho}(\mathrm{Top}^n_\star) \longrightarrow \operatorname{Ho}(\mathrm{Top}_\star)$ which are adjoint

$$\operatorname{Ho}(\mathrm{Top}_\star)(X^n, Y) \cong \operatorname{Ho}(\mathrm{Top}^n_\star)(X, Y).$$

Let $\operatorname{Ho}(\mathrm{Top}_\star)|_{(n-1)\text{-connected}}$ be the full subcategory of $\operatorname{Ho}(\mathrm{Top}_\star)$ determined by the $(n-1)$-connected spaces.

Theorem 3.2. *The pair of adjoint functors* $(\ )^n$, Id *induces an equivalence of categories*

$$\operatorname{Ho}(\mathrm{Top}^n_\star) \underset{Id}{\overset{(\ )^n}{\rightleftarrows}} \operatorname{Ho}(\mathrm{Top}_\star)|_{(n-1)\text{-connected}}.$$

Proof. It suffices to check that the unit and the counit of the adjunction are isomorphisms. This follows from Proposition 3.1.

Remarks. (i) The functor $(\ )^n\colon \operatorname{Ho}(\mathrm{Top}^n_\star) \longrightarrow \operatorname{Ho}(\mathrm{Top}_\star)$ preserves cofibration sequences and its right adjoint $Id\colon \operatorname{Ho}(\mathrm{Top}_\star) \longrightarrow \operatorname{Ho}(\mathrm{Top}^n_\star)$ preserves fibration sequences.

(ii) Let Σ^L , Ω denote the suspension and loop functors of the category $\operatorname{Ho}(\mathrm{Top}_\star)$, and σ^L_n , ω_n their analogues for the category $\operatorname{Ho}(\mathrm{Top}^n_\star)$; then we have the isomorphisms

$$(\sigma^L_n X)^n \cong \Sigma^L X^n \ , \ \Omega X \cong \omega_n X$$

in the categories $\operatorname{Ho}(\mathrm{Top}_\star)$ and $\operatorname{Ho}(\mathrm{Top}^n_\star)$, respectively.

(iii) Let A be an $(n-1)$-connected space and $f : X \longrightarrow Y$ be a map between $(n-1)$-connected spaces with homotopy fibre F_f in the category $\operatorname{Ho}(\mathrm{Top}_\star)$; then the following sequence is exact:

$$\cdots \to [A, (\Omega^2 F_f)^n] \to [A, (\Omega X)^n] \to [A, (\Omega Y)^n] \to [A, F^n_f] \to [A, X] \to [A, Y],$$

where $[\ ,\ ]$ denotes the hom-set in the category $\operatorname{Ho}(\mathrm{Top}_\star)$.

Next we compare the closed model category $\mathrm{Top}^n_\star$ with the closed model category SS_n. For the adjoint functors

$$\mathrm{SS}_n \underset{R_n\,\mathrm{Sing}}{\overset{|\ |}{\rightleftarrows}} \mathrm{Top}^n_\star$$

we have the following results:

Proposition 3.3. *Let $f\colon X \longrightarrow Y$ be a map in $\mathrm{Top}_\star$. Then*

(i) *f is an n-fibration if and only if $R_n\,\mathrm{Sing}\, f$ is a fibration in SS_n.*

(ii) *f is a weak n-equivalence if and only if $R_n\,\mathrm{Sing}\, f$ is a weak equivalence in SS_n.*

Proof. (i) The result is obtained by using the pairs of adjoint functors $|\ |, \mathrm{Sing}$ and $(\ )_{(n)}, R_n$ and taking into account that a fibration $g\colon K \longrightarrow L$ in SS_n, when L is a Kan complex, is characterized in [8, Proposition 2.12] by the RLP with respect to the inclusions $\mathrm{V(p,k)} \longrightarrow \triangle[p],\ p > n,\ 0 \leq k \leq p$.

(ii) Note that for $q \geq n$, $\pi_q(R_n\,\mathrm{Sing}\, Z) \cong \pi_q Z$ and for $q < n$, $\pi_q(R_n\,\mathrm{Sing}\, Z) \cong 0$.

Because for a map f in SS_n, f is a weak equivalence in SS_n if and only if $|f|$ is a weak n-equivalence, it follows that the functors $|\ |$ and $R_n\,\mathrm{Sing}$ induce adjoint functors in the respective categories of fractions:

$$\mathrm{Ho}(\mathrm{SS}_n) \underset{R_n\,\mathrm{Sing}}{\overset{|\ |}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Top}^n_\star).$$

Checking that the unit and the counit of the adjunction are isomorphisms, then one has:

Theorem 3.4. *The realization functor and the n-reduction of the singular functor induce the following equivalence of categories:*

$$\mathrm{Ho}(\mathrm{SS}_n) \underset{R_n\,\mathrm{Sing}}{\overset{|\ |}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Top}^n_\star).$$

Remarks. (i) Let $\pi\mathrm{CW}_n$ denote the category of pointed CW-complexes whose $(n-1)$-skeleton consists just of one 0-cell and the morphisms are given by pointed homotopy classes of pointed maps. Then the above functors induce an equivalence between the categories $\mathrm{Ho}(\mathrm{Top}^n_\star)$ and $\pi\mathrm{CW}_n$.

(ii) The localized category $\mathrm{Ho}(\mathrm{Top}^1_\star)$ is equivalent to the localized category of simplicial groups $\mathrm{Ho(SG)}$.

(iii) The localized category $\mathrm{Ho}(\mathrm{Top}^n_\star)$ is equivalent to the localized category of $(n-1)$-reduced simplicial groups $\mathrm{Ho}(\mathrm{SG}_{n-1})$.

(iv) We can combine the notion of weak equivalence for m-types given in [2] and the notion of weak equivalence given here to give algebraic models for spaces with nontrivial homotopy groups between n and m ($n \leq m$.) There are many algebraic models closely connected with simplicial groups for these spaces; see [1], [4].

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