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A NON-TREELIKE CONTINUUM THAT IS NOT THE 2-TO-1 IMAGE OF ANY CONTINUUM

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ABSTRACT. Some thirteen years ago S. B. Nadler, Jr. and L. E. Ward, Jr., asked if any treelike continuum could be the 2-to-1 image of a continuum. In fact, it has been conjectured that the property of being treelike characterizes those continua that are not the 2-to-1 image of any continuum. But the characterization must be something else; this paper shows that many pseudo-solenoids are not the 2-to-1 image of any continuum.

1. INTRODUCTION

The conjecture [8] that a continuum is the 2-to-1 image of a continuum if and only if it is not treelike is not true. Since the Nadler-Ward question described in the abstract was raised in 1983, it has been shown that many types of treelike continua are not 2-to-1 images of continua and no one has found one that is. See [8] for a description of results on this half of the conjecture. However, we will show in Section 1 that pseudo-solenoids with infinitely many bonding maps of even degree cannot be the 2-to-1 image of any continuum. This contrasts with a construction in [7] of a 2-to-1 covering map onto the planar pseudo-circle, an example of a pseudo-solenoid whose bonding maps do not have even degree. In [4] W. Dębski proved, using strongly the group structure of the solenoid, that there is no 2-to-1 map defined on a solenoid if infinitely many of its bonding maps are even. Although Dębski's result sounds similar, he was working with 2-to-1 domains; in fact, in [5] it was shown that every solenoid is a 2-to-1 retract of a continuum. In Section 2 there are theorems concerning when non-treelike continua are 2-to-1 retracts of continua (in a nutshell: almost always if the continuum is not hereditarily indecomposable). The known results at this point leave open the following questions:

Question 1. Does there exist a non-treelike continuum that is not hereditarily indecomposable and is not a 2-to-1 retract of any continuum?

Question 2. (The big question.) Exactly which continua are 2-to-1 images of continua?

By *continuum* we mean a connected compact metric space. Other definitions are in a glossary just before the bibliography.

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2. An example of a non-treelike continuum that is not the 2-to-1 image of any continuum

The construction of pseudo-solenoids was first described by J. T. Rogers, Jr. in his dissertation [11]. In that document he called all hereditarily indecomposable continua that are circularly chainable, but not chainable, "pseudo-circles". But now these continua are called "pseudo-solenoids" (even by Rogers). Rogers provided a systematic construction consisting of an inverse limit system on circles with essential, individually simplicial bonding maps. Pseudo-solenoids result if the maps are complicated enough, and he showed that all pseudo-solenoids have this structure. The definition of degree used in Theorem 1 can also be found in this paper, [11], although very little of the complexity of the definition is needed here.

Although I am sure that every English schoolgirl knows that pseudo-solenoids are not treelike, I was not able to find this fact in the literature; so Lemma 1 provides a proof. Note that the proof of Lemma 2 establishes the slightly stronger fact that the map g in question is a crisp map (see definition in the glossary); we use 2-fold covering map because that is all that is needed in the proof of Theorem 1 and because covering maps are better known than crisp maps.

Lemma 1. *No pseudo-solenoid is treelike.*

Proof. We will use Eilenberg's theorem (see 12.38 in [9]) that says that any continuous map f from a compact metric space Y into S^1 is inessential if and only if there is a map g from Y into the reals $\mathcal{R}$ such that $f = exp \circ g$, where exp is the map defined by $exp(t) = (\cos(t), \sin(t))$, for each real number t.

Let Y be a pseudo-solenoid; then by [11] $Y = \overleftarrow{\{Y_i, f_i\}}$, where each Y_i is the unit circle. We will show that the first projection, π, that sends each point of Y to its first coordinate in Y_1, is essential. Since Y is one-dimensional, it will follow from the Case-Chamberlin characterization of treelike continua [2] that Y is not tree like.

Suppose that K is a subset of the jth factor space Y_j. For this proof and the proof of Theorem 1, we will use the $\breve{K}$ notation, K *enlarged*, as follows:

$$\breve{K} = (\prod_{i=1}^{j-1} Y_i \times K \times \prod_{i=j+1}^{\infty} Y_i) \cap Y.$$

That is, $\breve{K}$ is the set of points in Y whose jth coordinate lies in K.

From the Eilenberg theorem, if π is not essential, then there is a map g from Y into the reals, $\mathcal{R}$, such that $\pi = exp \circ g$. There is a chain of open intervals $\mathcal{U} = \{U_1, U_2, ..., U_k\}$ covering the image $g(Y)$ in $\mathcal{R}$ whose links are small enough that the exp map is one-to-one on each U_i. Then there is an integer m large enough that if z is a point in the mth circle Y_m, then $g(\breve{z})$ is a subset of an element of $\mathcal{U}$. This means that every point of the set $\breve{z}$ maps to the same point in $\mathcal{R}$ under g, since each point in $\breve{z}$ has the same first coordinate.

For this integer m, define $f_{1,m} = f_1 \circ f_2 \circ ... \circ f_m$, an essential map from Y_m to Y_1, and define the map h from Y_m to $\mathcal{R}$ by $h(z) = g(\breve{z})$. The function is well defined since, as was explained above, g maps the set $\breve{z}$ to a single real number. It is straightforward to see that h is continuous, and the diagram commutes: $f_{1,m} = exp \circ h$. Thus, by Eilenberg's theorem again, $f_{1,m}$ is not essential. $\square$

Lemma 2. *If g maps the continuum X exactly 2-to-1 onto a pseudo-solenoid, then g is a 2-fold covering map.*

Proof. Lemma 2 in [6] states that if g is a 2-to-1 map from the continuum X onto the hereditarily indecomposable continuum Y, then g has a crisp restriction, and hence [6] a restriction that is a 2-fold covering map on a subcontinuum S of X. The pseudo-solenoid is hereditarily indecomposable and circularly chainable, so each proper subcontinuum is hereditarily indecomposable and chainable; hence by Bing's result [1] each proper subcontinuum of a pseudo-solenoid is a pseudo-arc. Since the restriction of g is 2-to-1, it cannot map onto Y (unless $S = X$) and so the image of the restriction is a pseudo-arc. But Theorem 3 in [6] states that there is no 2-to-1 map defined on a continuum whose image is a hereditarily indecomposable treelike continuum, and hence the image cannot be a pseudo-arc. Thus $S = X$ and the map g itself is a 2-fold covering map. $\square$

Lemma 3. *Suppose that g is a 2-fold covering map from the compact metric space X onto the compact metric space Y. Then there is an $\epsilon > 0$ such that every ϵ-chain of open sets in Y, $\{U_1, U_2, ..., U_k\}$, backs up under g^{-1} to two chains $\{V_1, V_2, ..., V_k\}$ and $\{W_1, W_2, ..., W_k\}$ in X whose unions are disjoint and such that g maps each of V_i and W_i homeomorphically onto U_i for each $i = 1, 2, ..., k$.*

Proof. Since g is locally one-to-one, there is a positive number δ such that if x and z are distinct elements of X and $g(x) = g(z)$, then $d(x, z) > 3\delta$. We will use the notation $N_\theta(t)$ to represent the θ neighborhood about the point t.

For each point y in Y, there is a positive number $\epsilon(y)$ such that if x and z denote the two points of $g^{-1}(y)$, then $g^{-1}(N_{\epsilon(y)}(y))$ is the union of two disjoint open sets, $E_x(y)$, a subset of $N_\delta(x)$, and $E_z(y)$, a subset of $N_\delta(z)$, and g maps each of $E_x(y)$ and $E_z(y)$ homeomorphically onto $N_{\epsilon(y)}(y)$. Since Y is compact, there is a single positive number, 2ϵ, that works for every $y \in Y$. Now, suppose that $\{U_1, U_2, ..., U_k\}$ is an chain of open sets in Y whose links have diameter no more than ϵ, and for each i, let y_i be a point in U_i. Denote by V_1 the subset $E_x(y_1) \cap g^{-1}(U_1)$ of X and denote by W_1 the subset $E_z(y_1) \cap g^{-1}(U_1)$ of X. The properties of ϵ and δ not only ensure that V_1 does not intersect W_1, they also ensure that neither of the two inverse sets V_1 or W_1 can intersect both of the next two inverse sets $E_x(y_2) \cap g^{-1}(U_2)$ and $E_z(y_2) \cap g^{-1}(U_2)$. But V_1 and W_1 each must intersect at least one of the two latter sets. Accordingly, denote by V_2 whichever of $E_x(y_2) \cap g^{-1}(U_2)$ and $E_z(y_2) \cap g^{-1}(U_2)$ intersects V_1 and denote by W_2 the other. Continue naming in this way and the two chains will be identified. $\square$

Theorem 1. *Suppose that Y is a pseudo-solenoid whose inverse limit representation has infinitely many bonding maps with even degree. Then there is no 2-to-1 map from any continuum onto Y.*

Proof. Suppose on the contrary that Y is a pseudo-solenoid that satisfies the hypothesis and g is a continuous 2-to-1 function from a continuum X onto Y. By Lemma 2, we know that g is a 2-fold covering map. From [11], $Y = \overleftarrow{\{Y_i, f_i\}}$, where each Y_i is the unit circle, each individual f_i is a simplicial map and, by hypothesis, infinitely many of the f_i have even degree.

For the map g from X to Y there is an $\epsilon > 0$ that satisfies the statement of Lemma 3. Then, there is a positive integer m such that f_m has even degree and there is a circular chain $\{U_1, U_2, ..., U_k\}$ of intervals covering the unit circle Y_m such that each enlarged link, $\breve{U}_i$, has diameter less than ϵ. (The notation $\breve{U}_i$ was defined in the proof of Lemma 1.) Then $\{\breve{U}_1, \breve{U}_2, ..., \breve{U}_k\}$ is a circular ϵ-chain covering Y.

By Lemma 3, the ϵ-chain $\{\breve{U}_1, \breve{U}_2, ..., \breve{U}_{k-1}\}$ (all but the last link) backs up under g^{-1} to two disjoint open chains $\{V_1, V_2, ..., V_{k-1}\}$ and $\{W_1, W_2, ..., W_{k-1}\}$ such that g maps each of V_i and W_i homeomorphically onto $\breve{U}_i$. Each proper subchain of the circular chain in Y backs up in this way, and upon reflection one sees that the entire circular chain either backs up to two disjoint circular chains (in case V_k intersects V_1 and W_k intersects W_1) or to one long circular chain (in case V_k intersects W_1 and W_k intersects V_1). Since X is connected, the latter case must hold. Notice that g follows the familiar 2-fold pattern of mapping the ith and $(k+i)$th links of the circular chain in X homeomorphically onto the ith link of the circular chain in Y, for $i = 1, 2, ..., k$.

Choose a point p in Y_m that lies in U_1. We will use the definition of degree from Rogers [11]. Since f_m is a simplicial map on the unit circle Y_{m+1}, there are only finitely many components of $f_m^{-1}(p)$ and their endpoints can be labeled $\{e_1, e_2, ..., e_n\}$ in order on Y_{m+1}. We will temporarily define the *parity of* f_m to be the parity of the number of intervals (e_i, e_{i+1}) such that f_m restricted to (e_i, e_{i+1}) maps onto Y_m, including possibly the interval (e_n, e_1). Note that the number of such intervals is not the same as the degree of the map which attaches $+1$ to those intervals that map onto Y_m in one direction and attaches -1 to those intervals that map onto Y_m in the other direction. Nevertheless, the parity of the function f_m is even since its degree is even.

There is a circular chain covering the circle Y_{m+1} whose links are small enough that (1) their images under f_m refine the circular chain $\{U_1, U_2, ..., U_k\}$ covering Y_m, (2) no link contains two points of $\mathcal{E} = \{e_1, e_2, ..., e_n\}$, and (3) every link containing each e_i is mapped by f_m into U_1. Because we are primarily interested in the links of this circular chain in Y_{m+1} that contain points of $\mathcal{E}$, we will effectively ignore the other links by labeling the circular chain $\{D_1, ..., D_2, ..., ..., D_n, ...\}$, where $e_i \in D_i$ for each $i = 1, 2, ..., n$. Then the enlarged circular chain $\mathcal{D} = \{\breve{D}_1, ..., \breve{D}_2, ..., ..., \breve{D}_n, ...\}$ is still an ϵ-circular chain covering Y that refines the first one, $\{\breve{U}_1, \breve{U}_2, ..., \breve{U}_k\}$. Notice, for each $i = 1, 2, ..., n$, how the enlarged sets $\breve{e}_i$ back up: $g^{-1}(\breve{e}_i) = E_i \cup F_i$, two disjoint compacta in X, labeled so that $E_i \subset V_1$ and $F_i \subset W_1$. Furthermore, each subchain $\{\breve{D}_i, ..., \breve{D}_{i+1}\}$ of $\mathcal{D}$ backs up under g^{-1} to two disjoint chains. One, $\mathcal{B}_i$, is contained either in $\{V_1, V_2, ..., V_k, W_1\}$ or $\{V_1, W_k, W_{k-1}, ..., W_1\}$, but either way its first link contains E_i and is contained in V_1. The other, $\mathcal{C}_i$, is contained either in $\{W_1, W_2, ..., W_k, V_1\}$ or $\{W_1, V_k, V_{k-1}, ..., V_1\}$, and its first link contains F_i and is contained in W_1. (This is also true for the subchain $\{\breve{D}_n, ..., \breve{D}_1\}$.) Two things can happen; if f_m restricted to the interval (e_i, e_{i+1}) maps *onto* Y_m, then the last link of $\mathcal{B}_i$ will contain F_{i+1} and will be contained in W_1, and the last link of $\mathcal{C}_i$ will contain E_{i+1} and will be contained in V_1. If, on the other hand, f_m restricted to the interval (e_i, e_{i+1}) is not onto, then there is no switch from E to F and back. Rather the last link of $\mathcal{B}_i$ will contain E_{i+1} and will be contained in V_1, and the last link of $\mathcal{C}_i$ will contain F_{i+1} and will be contained in W_1.

We will build with these pieces two circular chains, $\mathcal{G}$ and $\mathcal{H}$, in X whose unions are disjoint and cover X. This contradiction to the connectivity of X will complete the proof. The sets of chains $\{\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n\}$ and $\{\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_n\}$ will be divided into two camps to form the new circular chains, starting with $\mathcal{B}_1$ in $\mathcal{G}$ and $\mathcal{C}_1$ in $\mathcal{H}$. Where $\mathcal{B}_2$ and $\mathcal{C}_2$ go depends on f_m. If f_m restricted to the interval (e_1, e_2) in Y_{m+1} maps onto Y_m, then there is a switch in letters: $\mathcal{B}_2$ goes to $\mathcal{H}$ and $\mathcal{C}_2$ goes to $\mathcal{G}$. Otherwise, if f_m is not onto, there is no switch: $\mathcal{B}_2$ goes to $\mathcal{G}$ and $\mathcal{C}_2$ goes to $\mathcal{H}$. Either way, the first two chains in each of $\mathcal{G}$ and $\mathcal{H}$ will link up correctly at E_2 and F_2. So the general rule is this. If f_m restricted to the interval (e_i, e_{i+1}) in Y_{m+1} maps onto Y_m, then the chain $\mathcal{B}_{i+1}$ is concatenated onto the already-assigned chain $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ is concatenated onto $\mathcal{B}_i$; that is, there is a switch in letters. On the other hand, if f_m is not onto, there is no switch in letters; the chain $\mathcal{B}_{i+1}$ is concatenated onto the already-assigned chain $\mathcal{B}_i$ and $\mathcal{C}_{i+1}$ is concatenated onto $\mathcal{C}_i$.

Now, because the parity of f_m is even, there are an even number of switches from $\mathcal{B}$ to $\mathcal{C}$ and back in the constructions of each of $\mathcal{G}$ and $\mathcal{H}$. This means that $\mathcal{G}$ starts with E_1 in its first link, a link from the chain $\mathcal{B}_1$, and ends with E_1 in its last (equal to its first) link from the chain $\mathcal{B}_n$. Similarly, $\mathcal{H}$ starts with F_1 in its first link, a link from the chain $\mathcal{C}_1$, and ends with F_1 in its last link from $\mathcal{C}_n$. So the chains $\mathcal{G}$ and $\mathcal{H}$ are the disjoint circular chains needed for the contradiction. □

3. Non-treelike continua that are 2-to-1 retracts of continua

If a non-treelike continuum Y is not hereditarily indecomposable, then there is probably a 2-to-1 retraction from a continuum onto Y. Some known theorems and the theorems in this section will explain the "probably" . For instance, Nadler and Ward [10] showed that if a continuum Y fails to be hereditarily unicoherent, then Y is a 2-to-1 retract (of a continuum). So if there is a simple closed curve in Y for instance, or a Warsaw circle, then Y is a 2-to-1 retract. Another example: it was shown in [5] that all solenoids are 2-to-1 retracts of continua; and note that solenoids are hereditarily unicoherent. We show here in Theorem 2 with a simple construction that if a continuum Y is a 2-to-1 retract, then so is *any* continuum that contains Y; this "superset" phenomenon greatly expands the set of examples of continua that are known to be 2-to-1 retracts. Then we show in Theorems 3 and 4 that each hereditarily decomposable non-treelike continuum is a 2-to-1 retract, but no hereditarily indecomposable continuum (treelike or not) is a 2-to-1 retract. Also we demonstrate in Theorem 5 how the existence of an essential map onto the unit circle with at least one connected point inverse guarantees a 2-to-1 retraction.

Theorem 2. *Suppose that the continuum Y is a 2-to-1 retract of a continuum, and suppose that $Y \subset Z$. Then Z is also a 2-to-1 retract of a continuum.*

Proof. Let r denote a 2-to-1 retraction from the continuum X onto Y. Let Z_1 and Z_2 denote two copies of Z, with the corresponding copies of Y named Y_1 and Y_2, and let X_1 denote a copy of X with Y_3 its copy of Y. We may assume that Z_1, Z_2, and X_1 are disjoint. Now let W denote the union of Z_1, Z_2, and X_1 with Y_1, Y_2, and Y_3 identified into a single copy, say Y_4, of Y. There is a natural 2-to-1 retraction of W onto Z_1 that uses r (or a copy of r) from X_1 onto Y_4 and matches $Z_2 \setminus Y_2$ with $Z_1 \setminus Y_1$. □

Theorem 3. *If Y is a hereditarily decomposable non-treelike continuum, then there is a continuum that retracts exactly 2-to-1 onto Y.*

Proof. This follows immediately from H. Cook's theorem [3] that all λ-dendroids are treelike. A λ-dendroid is a hereditarily decomposable and hereditarily unicoherent continuum. So if Y is a hereditarily decomposable non-treelike continuum, then it cannot be hereditarily unicoherent, and the conclusion of Theorem 3 follows from the Nadler-Ward result described in this section's opening paragraph. □

Theorem 4. *No hereditarily indecomposable continuum is a 2-to-1 retract of a continuum.*

Proof. Suppose that the hereditarily indecomposable continuum Y is a 2-to-1 retract of a continuum X. Let r denote the retraction. As was used earlier in this paper, any 2-to-1 map from a continuum onto a hereditarily indecomposable continuum has a crisp restriction to a continuum in the domain ([6]). And, also from [6], each crisp map is a 2- fold covering map. So the restriction is a 2-fold covering map from a subcontinuum A of X onto a subcontinuum B of both A and Y. Now, the connected set A is equal to $(A \setminus B) \cup B$, two disjoint sets with the second set closed. Hence there is a point p in B that is a limit point of $A \setminus B$. Let $\{p_i\}$ denote a sequence of points in $A \setminus B$ that converges to p. By the continuity of r, the sequence $\{r(p_i)\}$ converges to $r(p)$ which is p since r is a retraction. Note that $r(p_i) \neq p_i$ since the former is in B and the latter is in $A \setminus B$; hence there are, arbitrarily close to p, two points of A that map the same under r. This means that the restriction of r is not locally one-to-one and so cannot be a 2-fold covering map. This contradiction completes the proof. $\square$

Theorem 5. *Suppose g is an essential map from the continuum Y onto the unit circle S^1 such that for some point p in S^1, the inverse $g^{-1}(p)$ is connected. Then Y is a 2-to-1 retract of a continuum.*

Proof. Since the points of the unit circle S^1 are determined by their polar angle, we will simplify the notation by assuming that g maps Y onto $S^1 = (0, 2\pi]$, and we'll try to remember that 2π is a limit point at the 0 end.

Suppose now that $g^{-1}(2\pi) = M$ is a continuum in Y. Construct the space $Z \subset Y \times [0, 2\pi]$ by

$$Z = (Y \times \{0\}) \cup \{(y, g(y)) | y \in Y\}.$$

We will think of $Y \times \{0\}$ as the original space Y. The map that sends both $(y, 0)$ and $(y, g(y))$ to $(y, 0)$ is a 2-to-1 retraction from Z onto $Y \times \{0\}$. But, is Z a continuum? The continuity of g ensures that Z is compact. Suppose that $Z = A \cup B$, two disjoint open and closed sets. One of them, say A, contains the connected set $Y \times \{0\}$. If B does not intersect $M \times \{2\pi\}$, then there are angles α and β with $0 < \alpha < \beta < 2\pi$ such that all of the second coordinates of points of B lie in the interval $[\alpha, \beta]$. Define the natural projection π from $Y \times (0, 2\pi]$ down to $Y \times \{0\}$ by the obvious formula $\pi(y, \theta) = (y, 0)$. Then $\pi(B)$ is both open and closed in $Y \times \{0\}$ but does not contain $Y \times \{0\}$. This contradicts the fact that Y is connected. Thus the connected set $M \times \{2\pi\}$ intersects B and thus is a subset of B. Since B is separated from $M \times \{0\}$, there is an angle $\alpha > 0$ such that the second coordinate of any point of B is greater than α, and similarly, since A is separated from $M \times \{2\pi\}$, there is an angle $\gamma < 2\pi$ such that the second coordinate of any point of A is less than γ.

We will show that this structure makes the map g inessential; a contradiction that implies that Z must be connected. The space $Y \times \{0\}$ is the union of two closed sets, $B_1 = \pi(B)$ and $A_1 = \pi(A \cap (Y \times (0, 2\pi])) \cup (M \times \{0\})$, whose intersection is $M \times \{0\}$. We define two homotopies, H_1 on $B_1 \times [0, 1]$, and H_2 on $A_1 \times [0, 1]$, both into S^1, such that these two homotopies agree (in fact are constant) on the intersection, $M \times \{0\}$, of their domains and the two homotopies end with the same constant map. Their union is a homotopy from g to a constant map. For each

$t \in [0,1]$, define:

$H_1(b,t) = 2\pi t + (1-t)g(b)$ for $b \in B_1 \setminus (M \times \{0\})$ and $H_1(m,t) = 2\pi$ for $m \in M \times \{0\}$ and

$H_2(a,t) = (1-t)g(a)$ for $a \in A_1 \setminus (M \times \{0\})$ and $H_2(m,t) = 0$ for $m \in M \times \{0\}$. $\square$

4. Definitions

1. **Chain (Circular chain).** A *chain (circular chain)* of sets, called *links*, is a finite collection that can be indexed $\{S_1, S_2, ..., S_k\}$ so that S_i intersects S_j if and only if $|i-j| \leq 1$ (except that for circular chains S_1 intersects S_k).
2. **Chainable (Circularly chainable).** A continuum is *chainable (circularly chainable)* if for each $\epsilon > 0$ there is an ϵ-chain (ϵ-circular chain) of open sets covering the continuum.
3. **Continuum.** A topological space is a *continuum* if it is connected, compact, and metric.
4. **Crisp.** A map f is *crisp* if, for each proper subcontinuum C in the image, there are exactly two components of the preimage of C and f maps each of these components homeomorphically onto C.
5. **Degree of a map.** For the definition of the degree of a simplicial map from S^1 onto itself, see [11].
6. **Essential map.** A map is *essential* if it is not homotopic to a constant map.
7. **ϵ-chain.** A chain is an *ϵ-chain* if each link has diameter less than ϵ. And the same holds for ϵ-circular chain.
8. **ϵ-map.** An *ϵ-map* is a continuous function whose point inverses have diameter less than ϵ.
9. **Pseudo-solenoid.** A continuum is a *pseudo-solenoid* if it is hereditarily indecomposable and circularly chainable but not chainable.
10. **Solenoid.** A continuum that is an inverse limit of circles such that each bonding map is an n-fold covering map for some integer n. A solenoid that is not a circle is indecomposable and each proper nondegenerate subcontinuum of any solenoid is an arc.
11. **2-to-1.** A function is *2-to-1* if the preimage of each point in the image has exactly two points.
12. **Treelike.** A continuum is *treelike* is for each $\epsilon > 0$, there is an ϵ-map from the continuum onto a tree (an acyclic graph).

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